Supplemental Appendices to: "Robust Real Rate Rules"

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Appendix H Other solutions to the ZLB

H.1 Equilibrium selection with perpetuities

The modified inflation target real rate rules of Section 4 of the main text delivered uniqueness conditional on a terminal condition ruling out inflation explosions or permanent ZLB episodes. In this subappendix, we examine how these two classes of undesirable equilibria may be avoided. This will enable us to answer Cochrane's (2011) argument that there is nothing to rule out non-stationary equilibria under monetary rules satisfying the Taylor-principle, and Benhabib, Schmitt-Grohé & Uribe's (2001) argument that there is nothing to rule out permanent ZLB spells under such rules.

We suppose that perpetuities (also called "consols") are traded in the economy. While actual perpetuities are rare, households may be able to approximate the flow of coupons from a perpetuity via holding a portfolio of government debt of different maturities. Additionally, there are many regular transfers from government to households or firms, such as unemployment benefits. While it is hard for households to capitalize and trade their flow of unemployment benefits, long-term government contracts (in defence, aerospace, etc.) certainly can be capitalized and traded. As long as such contracts enable a flow of nominal firm profits, their value will have a perpetuity-like component.

Perpetuity prices are functions of the entire expected future path of nominal rates, and hence they embed information on the economy's selected equilibrium. Crucially, if the economy is stuck at the ZLB, then perpetuity prices will be extremely high, or even infinite. For the sake of exposition, we will derive results for the more general class of geometric coupon bonds, and later specialise to the perpetuity case.

We assume that one unit of the period t geometric coupon bond bought at t returns \$1 at $t+1$, along with $\omega \in (0,1]$ units of the period $t+1$ geometric coupon bond. The $\omega = 1$ case corresponds to a perpetuity. The geometric coupon bond trades at a price of Q_t at t. Thus, if Ξ_{t+1} is the real SDF between periods t and $t+1$, and $\Pi_{t+1} := \exp \pi_{t+1}$ is gross inflation between these periods, then the price of the bond must satisfy:

$$Q_t = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [\omega Q_{t+1} + 1].$$

We assume that the government and central bank are the only institutions trusted enough to issue geometric coupon bonds, since private companies generally have shorter lives than nations. Thus, the total stock of such bonds, B_t, is in the government and/or central bank's control. We assume there is some $\underline{B} > 0$ such that in all states of the world $B_t \geq \underline{B}\omega^t$. For this it is enough that the government issued geometric coupon bonds at some point in the past, with the commitment to never buy all of them back. Since it is optimal for governments to fund themselves with perpetuities (Debortoli, Nunes & Yared 2017; 2022), this does not seem an unreasonable commitment. Then, the household's period t transversality condition on geometric coupon bond holdings states that:

$$0 = \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] Q_{t+s} B_{t+s} \geq \underline{B} \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] Q_{t+s} \omega^{t+s} \geq 0,$$

and hence $\lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \omega^s Q_{t+s} = 0$. Thus, for all t:

$$Q_t = \mathbb{E}_t \sum_{s=1}^{\infty} \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \omega^{s-1} = \mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=0}^{s} \frac{1}{I_{t+k}} \right] \omega^s,$$

where, as usual, I_t is the gross interest rate on a one period nominal bond (so $I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1$).

Now suppose that $I_{t+k} = 1$ (with probability one, conditional on period t

information) for all sufficiently high k. Then $Q_{t+s} = \frac{1}{1-\omega}$ (with conditional probability one) for all sufficiently high s. So, the transversality condition holds if and only if:

$$0 = \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \frac{\omega^s}{1-\omega} = \lim_{s\to\infty} \frac{\omega^s}{1-\omega},$$

i.e., if and only if $|\omega| < 1$. In particular, it is violated if the bond is a perpetuity, meaning $\omega = 1$.[1]

In other words, permanent stays at the ZLB do in fact violate a transversality constraint when the stock of perpetuities is positive. Intuitively, with households having infinite nominal wealth, they wish to spend some of that wealth today on real goods, which ends up violating (real) goods market clearing. The only way goods market clearing could be restored is if inflation is infinite when nominal wealth is. We show this carefully in Appendix K.12 below. However, under standard assumptions on money demand, infinite inflation is only possible with infinite money supply growth, which is likely to be physically impossible for a central bank. Infinite inflation is also ruled out by arbitrarily small degrees of price stickiness. Thus, as long as infinite inflation is ruled out by these considerations or some other, there is no equilibrium with a permanent ZLB stay.[2]

We now use this fact to construct a monetary rule with both global uniqueness and local determinacy, the latter helping ensure learnability. We assume that the central bank sets nominal interest rates via a tweaked non-linear version of the modified inflation target real rate rule of Section 4 of the main text. Our first tweak is that for simplicity, we assume that the inflation target is set one period in advance. Our second tweak is to introduce "punishment" in the form of a switch to the ZLB following large deviations. To define a large deviation, we will introduce an upper bound $\bar{I} > 1$ on gross nominal interest rates, and we will

[1] The necessity of the transversality constraint is non-obvious in the $\omega = 1$ case. However, in Appendix K.12 below we show that the problem with perpetuities can be transformed into a "cake eating" type problem with one period bonds, for which the transversality constraint is trivially necessary, even when $\omega = 1$.

[2] Government debt leading to a violation of the household transversality constraint at the ZLB may remind the reader of Benhabib, Schmitt-Grohé & Uribe (2002). However, the current proposal preserves fully-Ricardian fiscal policy (see Appendix K.12 below), and does not require the government to commit to take actions that are ex post undesirable (like increasing primary deficits in the face of exploding debt).

construct the modified inflation target to ensure gross nominal interest rates are strictly inside $(1, \bar{I})$ in equilibrium.

We suppose that the central bank sets:

$$I_t = \begin{cases} \max\left\{1, R_t \breve{\Pi}^*_{t+1|t} \left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi}\right\}, & \text{if } I_{t-1} \in (1, \bar{I}) \\ 1, & \text{otherwise} \end{cases},$$

where:

$$\breve{\Pi}^*_{t+1|t} := \max\left\{\frac{\mathcal{E}}{R_t}, \min\left\{\frac{\bar{I}}{\mathcal{E} R_t}, \Pi^*_{t+1|t}\right\}\right\},$$

with $\phi > 1$ and $\mathcal{E} := \exp \epsilon \in \left(1, \sqrt{\bar{I}}\right)$. It is easy to see that $\Pi_t = \breve{\Pi}^*_{t|t-1}$ for all t is consistent with this rule and the standard nominal and real bond pricing equations:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1, \qquad R_t \mathbb{E}_t \Xi_{t+1} = 1.$$

In the local vicinity of this equilibrium path, we have $I_t = R_t \breve{\Pi}^*_{t+1|t} \left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi}$, which implies:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\breve{\Pi}^*_{t+1|t}}{\Pi_{t+1}} = \left(\frac{\breve{\Pi}^*_{t|t-1}}{\Pi_t}\right)^{\phi}.$$

This has a unique stationary solution under mild conditions by the results of Online Appendix E.

To analyse potential deviations from this equilibrium, we switch to an economy without uncertainty for simplicity. This is in line with Cochrane (2011) which is also primarily concerned with deterministic economies.

First, suppose that for some reason, for some $t = t_0$, $\Pi_t > \breve{\Pi}^*_{t|t-1}$, but $I_{t-1} \in (1, \bar{I})$. Then:

$$\frac{\Pi_{t+1}}{\breve{\Pi}^*_{t+1|t}} = \max\left\{\frac{1}{R_t \breve{\Pi}^*_{t+1|t}}, \left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi}\right\} \geq \left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi},$$

and so $\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}$ explodes upwards as $t \to \infty$. Now for all t, $\breve{\Pi}^*_{t+1|t} \geq \frac{\mathcal{E}}{R_t}$, hence $\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}} \leq \frac{R_{t-1}\Pi_t}{\mathcal{E}} < I_{t-1}$. Thus, I_t must also (start to) explode upwards as $t \to \infty$. So, eventually, for some $t_1 \geq t_0$, $I_{t_1} > \bar{I}$. Thus $I_{t_1+1} = I_{t_1+2} = \cdots = 1$ according to the monetary rule. But this is only consistent with household optimality if Π_t is infinite at least once in $[t_0, \ldots, t_1]$, which in turn is physically impossible. Hence, there is no equilibrium with such a deviation.

Now, suppose that for some reason, for some $t = t_0$, $\Pi_t < \breve{\Pi}^*_{t|t-1}$, but $I_{t-1} \in (1, \bar{I})$. Then:

$$\frac{\Pi_{t+1}}{\breve{\Pi}^*_{t+1|t}} = \max\left\{\frac{1}{R_t\breve{\Pi}^*_{t+1|t}}, \left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi}\right\},$$

and so $\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}$ either explodes downwards towards zero forever as $t \to \infty$ or hits $I_{t_1} = 1$ at some $t_1 \geq t_0$. Now for all t, $\breve{\Pi}^*_{t+1|t} \leq \frac{\bar{I}}{\mathcal{E}R_t}$, hence $\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}} \geq \frac{\mathcal{E}R_{t-1}\Pi_t}{\bar{I}} = \frac{\mathcal{E}}{\bar{I}} I_{t-1}$. Thus, in fact I_t must hit $I_{t_1} = 1$ at some $t_1 \geq t_0$. Thus, just as before, $I_{t_1+1} = I_{t_1+2} = \cdots = 1$, which is inconsistent with equilibrium, ruling out the initial deviation.

Therefore, if households hold perpetuities, this tweaked real rate rules succeeds in producing global uniqueness. Admittedly, the punishment reduces its robustness, but for moderately high $\mathcal{E}$ and $\bar{I}$, and high ϕ, accidentally falling into the punishment regime would be very unlikely, even with additional uncertainty coming from wedges in the Fisher equation.

Of course, if there is something else in the economy ruling out explosive paths for inflation, then the punishment regime is unnecessary, and the central bank could just use the rule:

$$I_t = \max\left\{1, R_t\breve{\Pi}^*_{t+1|t}\left(\frac{\Pi_t}{\breve{\Pi}^*_{t|t-1}}\right)^{\phi}\right\}, \qquad \breve{\Pi}^*_{t+1|t} := \max\left\{\frac{\mathcal{E}}{R_t}, \Pi^*_{t+1|t}\right\}.$$

With households holding perpetuities, this still has no equilibria that are permanently stuck at the ZLB. Sticky prices are sufficient to rule out explosive equilibria, both as inflation is bounded above under standard price stickiness specifications (see Online Appendix E.1), and because under sticky prices, exploding inflation implies exploding real costs of this inflation. While prices may become more flexible at high inflation rates, there are practical limits on how often prices can change even under extreme hyperinflation. The price must at least remain constant for the time between picking an item off the shelf and arriving with it at the check-out. If this is correct, then even without a punishment regime, trade in perpetuities is sufficient to ensure a unique long-run equilibrium with inflation at target.

H.2 Price level real rate rules

One way to improve the performance of real rate rules near the ZLB is to replace the response to inflation with a response to the price level. Holden (2023) shows that responding to the price level is a robust way to ensure the existence of a unique solution with the ZLB, at least given that inflation does not converge to the deflationary steady state. We discussed how to rule out convergence to the deflationary steady state in the previous subappendix.

Price level rules rule out self-fulfilling temporary jumps to the ZLB as under a price level rule, the deflation during the bound period must be made up for by high inflation after exiting the bound. Thus, expected inflation is high in the last period at the bound, which via the Fisher equation, implies nominal interest rates should be high that period as well, unless real rates are still very low. This unwinds non-fundamental ZLB spells, as in a non-fundamental jump to the bound, real rates are unlikely to move enough to drive the economy to the ZLB on their own.

Incorporating the ideas from the Subsection 4.2, a variable target price level real rate rule takes the form:

$$i_t = \max\{0, r_t + \mathbb{E}_t \breve{p}^*_{t+1} - \breve{p}^*_t + \theta(p_t - \breve{p}^*_t)\},$$

with:

$$\breve{p}^*_t = \breve{p}^*_{t-1} + \max\{(1-\varrho)(p^*_t - \breve{p}^*_{t-1}) + \varrho(p^*_t - p^*_{t-1}), \epsilon - r_{t-1}\},$$

where p_t is the logarithm of the price level (so $\pi_t = p_t - p_{t-1}$),[3] p^*_t is the price level target, $\theta > 0$ controls the response to price deviations, $\epsilon > 0$ is a small constant and $\varrho \in [0,1)$ controls the speed with which $\breve{p}^*_t$ returns to p^*_t following a constrained spell. (Some of our results will require ϱ to be sufficiently close to 1, so price level gaps are not closed too quickly.) This has a solution in which $p_t = \breve{p}^*_t$ for all t, since if this holds, then from the monetary rule:

$$i_t - r_t = \max\{-r_t, \mathbb{E}_t \breve{p}^*_{t+1} - \breve{p}^*_t\} = \mathbb{E}_t \breve{p}^*_{t+1} - \breve{p}^*_t = \mathbb{E}_t \pi_{t+1},$$

as $\mathbb{E}_t \breve{p}^*_{t+1} - \breve{p}^*_t \geq \epsilon - r_t > -r_t$ by the definition of $\breve{p}^*_{t+1}$, so the Fisher equation holds as required.

[3] Note that there is no reason linearised NK models expressed in terms of the price level should be less accurate than linearised NK models expressed in terms of inflation. First, note that the equation $\pi_t = p_t - p_{t-1}$ holds exactly, as it results from taking logarithms of the equation $\Pi_t = \frac{P_t}{P_{t-1}}$, where Π_t is gross inflation and P_t is the price level. Secondly, note that as long as the model's equations can be expressed in terms of inflation, not the price level, prior to linearisation (something which is true in virtually all NK models), then they will be accurate as long as inflation is near steady state, even if the price level is far from its path in the absence of shocks. We do not impose that prices should be stationary, only that inflation is.

Note that $\theta > 0$ is sufficient for determinacy in the absence of the ZLB, since then the monetary rule and Fisher equation imply that:[4]

$$\mathbb{E}_t(p_{t+1} - \breve{p}^*_{t+1}) = (1+\theta)(p_t - \breve{p}^*_t).$$

Thus, price level rules have the same advantage of smoothed rules in not requiring $\phi > 1$. Convincing agents that $\theta > 0$ is likely easier than convincing them that $\phi > 1$, as argued in Subsection 2.1. Furthermore, just like standard (inflation) real rate rules, price level real rate rules are robust, since away from the bounds, price level determination is completely independent of the real interest rate or the rest of the model. Their chief advantage over standard real rate rules is in avoiding the multiplicity of transition paths highlighted by Holden (2023). In fact, Holden (2023) shows that in standard models they would avoid perfect foresight multiplicity and non-existence problems even had we set $\breve{p}^*_t := p^*_t$. Nonetheless, our definition of $\breve{p}^*_t$ gives additional robustness, as we will now show by replicating the arguments and conclusions of Subsection 4.3 of the paper and Appendix K.9 below, with the price level real rate rule in place of the smoothed real rate rule.

Uniqueness conditional on the modified target. Closely following Appendix K.9 below, we want to prove uniqueness of equilibrium under our price level real rate rule (introduced in period 1), without uncertainty, and assuming that $p_{t+1} - p_t$ and $\breve{p}^*_{t+1} - \breve{p}^*_t$ are bounded in t, and that the economy eventually escapes the ZLB for good. The latter assumption implies there must exist a smallest possible $s \geq 1$ such that for all $t \geq s$, the ZLB does not bind. We assume for a contradiction that $s > 1$, hence for all $t \geq s$, by the monetary rule and Fisher equation:[5]

$$r_t + p_{t+1} - p_t = i_t = r_t + \breve{p}^*_{t+1} - \breve{p}^*_t + \theta(p_t - \breve{p}^*_t),$$

meaning:

$$(p_{t+1} - \breve{p}^*_{t+1}) = (1+\theta)(p_t - \breve{p}^*_t),$$

so for $t \geq s$, $p_t - \breve{p}^*_t = (1+\theta)^{t-s}(p_s - \breve{p}^*_s)$, and hence $(p_{t+1} - p_t) - (\breve{p}^*_{t+1} - \breve{p}^*_t) = \theta(1+\theta)^{t-s}(p_s - \breve{p}^*_s)$. Since $(1+\theta)^{t-s} \to \infty$ as $t \to \infty$, this in turn implies that $p_s =$

[4] Note that explosions of $p_t - \breve{p}^*_t$ imply explosions of $(p_{t+1} - p_t) - (\breve{p}^*_{t+1} - \breve{p}^*_t)$, which are ruled out by our boundedness assumptions. For example, if $p_t - \breve{p}^*_t = (1+\theta)^t(p_0 - \breve{p}^*_0)$, then $(p_{t+1} - p_t) - (\breve{p}^*_{t+1} - \breve{p}^*_t) = \theta(1+\theta)^t(p_0 - \breve{p}^*_0)$.

[5] Note that we can drop expectations as there is no uncertainty.

$\breve{p}_s^*$, by our boundedness assumptions. But as the economy is at the ZLB in period $s-1$, $0 = i_{s-1} = r_{s-1} + p_s - p_{s-1} = r_{s-1} + (\breve{p}_s^* - \breve{p}_{s-1}) - (p_{s-1} - \breve{p}_{s-1}) \geq r_{s-1} + \epsilon - r_{s-1} - (p_{s-1} - \breve{p}_{s-1}) > -(p_{s-1} - \breve{p}_{s-1})$, meaning that $p_{s-1} - \breve{p}_{s-1} > 0$. Now, by the period $s-1$ monetary rule, $0 \geq r_{s-1} + \mathbb{E}_{s-1}\breve{p}_s^* - \breve{p}_{s-1}^* + \theta(p_{s-1} - \breve{p}_{s-1}^*) > r_{s-1} + \mathbb{E}_{s-1}\breve{p}_s^* - \breve{p}_{s-1}^* \geq r_{s-1} + \epsilon - r_{s-1} = \epsilon > 0$, giving the required contradiction. Thus $s = 1$, meaning the economy never hits the ZLB. Combined with the determinacy in the absence of the ZLB previously proven, this establishes the uniqueness of the $p_t = \breve{p}_t^*$ solution conditional on the path of $\breve{p}_t^*$.

Uniqueness of the modified target. Again closely following Appendix K.9 below, we also want to prove that in the model given by equations (10) and (11), from Subsection 4.1, there is a unique perfect foresight solution for $\breve{p}_t^*$. We assume that all exogenous processes are constant at their steady-state level, that $p_t^* = \pi^*(t-1)$, and that all variables are at steady-state in period 0 (relative to trend in the case of prices), since none of these assumptions have any impact on uniqueness, by the results of Holden (2023). (This also means that our results are robust to adding any shocks to the model.) We also impose that the ZLB never binds, since we have already established this under our retained assumptions. Given this, we replace the notation $\breve{p}_{t+1}^*$ with $\breve{p}_{t+1|t}^*$, since $\breve{p}_{t+1}^*$ is known in period t given that p_t^* is now deterministic. Likewise, we replace p_{t+1} with $p_{t+1|t}$, as $p_{t+1} = \breve{p}_{t+1}^* = \breve{p}_{t+1|t}^*$, known at t. Note that $p_{1|0} = \breve{p}_{1|0}^* = p_1^* = 0$. Finally, we define $\hat{p}_{t+1|t} := p_{t+1|t} - \pi^* t$ and $\hat{\breve{p}}_{t+1|t}^* := \breve{p}_{t+1|t}^* - \pi^* t$. This gives the following equations for $t \geq 1$:

$$\beta(\hat{p}_{t+1|t} - \hat{p}_{t|t-1}) + \kappa x_t = \begin{cases} 0, & \text{if } t = 1 \\ \hat{p}_{t|t-1} - \hat{p}_{t-1|t-2}, & \text{if } t > 1 \end{cases}$$

$$i_t = \begin{cases} r_t + \pi^* + \hat{\breve{p}}_{t+1|t}^*, & \text{if } t = 1 \\ r_t + \pi^* + \hat{\breve{p}}_{t+1|t}^* - \hat{\breve{p}}_{t|t-1}^* + \theta\left(\hat{p}_{t|t-1} - \hat{\breve{p}}_{t|t-1}^*\right), & \text{if } t > 1 \end{cases}{}'$$

$$x_t = \delta x_{t+1} - \varsigma(r_t - n), \qquad i_t = r_t + \pi^* + \hat{p}_{t+1|t} - \hat{p}_{t|t-1},$$

$$\hat{\breve{p}}_{t+1|t}^* = \max\left\{\varrho\hat{\breve{p}}_{t|t-1}^*, \hat{\breve{p}}_{t|t-1}^* + \epsilon - r_t - \pi^*\right\},$$

where we assume $\kappa\varsigma \neq 0$, $\theta > 0$ and $n + \pi^* > \epsilon > 0$. The latter assumption ensures that $\hat{\breve{p}}_{t+1|t}^* = 0$ in steady state.

We are interested in the constraint in the definition of $\hat{\breve{p}}_{t+1|t}^*$, which we note can be rewritten as the pair of equations:

$$z_t = \hat{\breve{p}}_{t+1|t}^* - \hat{\breve{p}}_{t|t-1}^* + r_t + \pi^* - \epsilon,$$

$$z_t = \max\{0, -(1-\varrho)\hat{\hat{p}}^*_{t|t-1} + r_t + \pi^* - \epsilon\},$$

where z_t is an auxiliary variable. The results of Holden (2023) imply that in order to prove uniqueness under perfect foresight (conditional on z_t eventually converging to its positive steady state value), we should first replace the second equation for z_t just given with $z_t = -(1-\varrho)\hat{\hat{p}}^*_{t|t-1} + r_t + \pi^* - \epsilon + y_t$, where y_t is an exogenous forcing process. For convenience, we define $y_t := 0$ for $t \le 0$. Then, for $t \ge 1$:

$$\hat{p}_{t+1|t} = \hat{\hat{p}}^*_{t+1|t} = \varrho\hat{\hat{p}}^*_{t|t-1} + y_t = \sum_{j=0}^{\infty} \varrho^j y_{t-k},$$

$$x_t = \frac{1}{\kappa}\left[-\beta y_t + [1 + (1-\varrho)\beta]y_{t-1} + \sum_{j=2}^{\infty}\big[[1 + (1-\varrho)\beta]\varrho - 1\big]\varrho^{j-2}y_{t-j}\right],$$

$$\begin{aligned} r_t = n + \frac{1}{\kappa\varsigma}\Bigg[&-\beta\delta y_{t+1} + \big[\beta + \delta[1 + (1-\varrho)\beta]\big]y_t \\ &- \big[\delta + (1-\delta\varrho)[1 + (1-\varrho)\beta]\big]y_{t-1} \\ &- \sum_{j=2}^{\infty}(1-\delta\varrho)\big[[1 + (1-\varrho)\beta]\varrho - 1\big]\varrho^{j-2}y_{t-j}\Bigg],\end{aligned}$$

$$\begin{aligned} z_t &= n + \pi^* - \epsilon + y_t - (1-\varrho)y_{t-1} - \sum_{j=2}^{\infty}\varrho(1-\varrho)\varrho^{j-2}y_{t-j} \\ &+ \frac{1}{\kappa\varsigma}\Bigg[-\beta\delta y_{t+1} + \big[\beta + \delta[1 + (1-\varrho)\beta]\big]y_t - \big[\delta + (1-\delta\varrho)[1 + (1-\varrho)\beta]\big]y_{t-1} \\ &\qquad - \sum_{j=2}^{\infty}(1-\delta\varrho)\big[[1 + (1-\varrho)\beta]\varrho - 1\big]\varrho^{j-2}y_{t-j}\Bigg],\end{aligned}$$

from, respectively, the monetary rule and Fisher equation, the equations for z_t, the Phillips curve, the Euler equation, and the first equation for z_t.

Holden (2023) shows that uniqueness is determined by the determinants of the principal sub-matrices of the "M" matrix for the model, which, here, contains the partial derivatives of z_t (t in rows) with respect to y_s (s in columns). We take M to have infinitely many rows and columns in the following. By our solution for z_t, M is a Toeplitz, lower Hessenberg matrix. The values on each of the diagonals of M may be read off from the solution for z_t. We assume for simplicity that $\beta > 0$, $\delta > 0$ and $\kappa\varsigma > 0$, which implies that $\frac{1}{\kappa\varsigma}\big[\beta + \delta[1 + (1-\varrho)\beta]\big] > 0$, so the diagonal elements are greater than one. We also assume that $(1-\beta)(1-\delta) - \kappa\varsigma < 0$, as in Appendix K.8, for example, and that $1 - \beta\delta > (1-\beta)(1-\delta) - \kappa\varsigma$, for which it is sufficient (but not necessary) that $1 - \beta\delta \ge 0$. Note for future reference that if $\varrho =$

1, then the M matrix is identical to the one in Appendix K.9.

Now consider a finite size principal sub-matrix of M. Since M is lower Hessenberg and Toeplitz, this sub-matrix must be block lower triangular, where each block on the diagonal is either lower triangular (with $1+\frac{1}{\kappa\varsigma}\left[\beta+\delta[1+(1-\varrho)\beta]\right]$ on the diagonal), or Hessenberg and Toeplitz, being a contiguous principal sub-matrix of M. Recall that the determinant of a block triangular matrix is the product of the determinants of the blocks on the diagonal. Thus, the sub-matrix will have determinant greater than one if each of the sub-matrix's blocks has determinant greater that one. Since $\frac{1}{\kappa\varsigma}\left[\beta+\delta[1+(1-\varrho)\beta]\right]>0$, a triangular block of size $S\times S$ has determinant of $\left(1+\frac{1}{\kappa\varsigma}\left[\beta+\delta[1+(1-\varrho)\beta]\right]\right)^S>1$. Thus, we just need to check the determinants of the Hessenberg and Toeplitz blocks, which are contiguous principal sub-matrices of M.

By the results of Cahill et al. (2002) (which were also used in Online Appendix H.3 of Holden (2023)), the determinant of any $S\times S$ Hessenberg and Toeplitz block is given by m_S, where:

$$m_{-1}:=m_{-2}:=\cdots=0,\qquad m_0:=1,$$

$$\begin{aligned}m_S=&\left[1+\frac{1}{\kappa\varsigma}\left[\beta+\delta[1+(1-\varrho)\beta]\right]\right]m_{S-1}\\&-\frac{\beta\delta}{\kappa\varsigma}\left[(1-\varrho)+\frac{1}{\kappa\varsigma}\left[\delta+(1-\delta\varrho)[1+(1-\varrho)\beta]\right]\right]m_{S-2}\\&-\left[\varrho(1-\varrho)+\frac{1}{\kappa\varsigma}(1-\delta\varrho)\left[[1+(1-\varrho)\beta]\varrho-1\right]\right]\sum_{k=2}^{\infty}\left(\frac{\beta\delta}{\kappa\varsigma}\right)^k\varrho^{k-2}m_{S-k-1}.\end{aligned}$$

Multiplying this last equation by the lag polynomial $I-\frac{\beta\delta}{\kappa\varsigma}\varrho L$, then gives the simpler expression:

$$m_S=\left[1+\frac{\beta+\delta+\beta\delta}{\kappa\varsigma}\right]m_{S-1}-\frac{\beta\delta}{\kappa\varsigma}\left[1+\frac{\beta+\delta+1}{\kappa\varsigma}\right]m_{S-2}+\frac{1}{\kappa\varsigma}\left(\frac{\beta\delta}{\kappa\varsigma}\right)^2m_{S-3},$$

which does not directly depend on ϱ (though ϱ will impact the initial conditions). As in Appendix K.9, let:

$$d:=\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^2-4\frac{\beta\delta}{(\kappa\varsigma)^2}=1+2\frac{\beta+\delta}{\kappa\varsigma}+\frac{(\beta-\delta)^2}{(\kappa\varsigma)^2}>1,$$

as $\frac{\beta+\delta}{\kappa\varsigma}>0$ by assumption. Additionally, from the fact that $\beta\delta>0$, we have that $1<d<\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^2$, so $1<\sqrt{d}<1+\frac{\beta+\delta}{\kappa\varsigma}$. Given the solution for m_S we found for the $\varrho=1$ case in Appendix K.9, the recurrence for m_S just derived implies that for some constants A, B and C:

$$\begin{aligned} m_S &= A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \frac{B}{2^S}\left(1+\frac{\beta+\delta}{\kappa\varsigma}+\sqrt{d}\right)^S + \frac{C}{2^S}\left(1+\frac{\beta+\delta}{\kappa\varsigma}-\sqrt{d}\right)^S \\ &= A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \frac{1}{2^S}\sum_{k=0}^{S}\binom{S}{k}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^k \sqrt{d}^{S-k}\left[B+C(-1)^{S-k}\right] \\ &= A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \frac{B+C}{2^S}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^S \\ &\qquad + \frac{1}{2^S}\sum_{k=0}^{S-1}\binom{S}{k}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^k \sqrt{d}^{S-k}\left[B+C(-1)^{S-k}\right]. \end{aligned}$$

Furthermore, the initial conditions imply that:

$$\begin{aligned} A &= \frac{(1-\varrho)\beta\delta}{(1-\beta)(1-\delta)-\kappa\varsigma}, \\ B &= \frac{1}{\sqrt{d}}\left[\frac{A}{\kappa\varsigma}+(1-A)\frac{1}{2}\left(1+\frac{\beta+\delta}{\kappa\varsigma}+\sqrt{d}\right)\right], \\ C = 1-A-B &= -\frac{1}{\sqrt{d}}\left[\frac{A}{\kappa\varsigma}+(1-A)\frac{1}{2}\left(1+\frac{\beta+\delta}{\kappa\varsigma}-\sqrt{d}\right)\right]. \end{aligned}$$

Note, that since $(1-\beta)(1-\delta)-\kappa\varsigma<0$ and $\beta\delta>0$, $\frac{\beta\delta}{(1-\beta)(1-\delta)-\kappa\varsigma}<0$, and so $A<0$ as $\varrho\in[0,1)$. Thus, $B+C=1-A>1$. Furthermore:

$$\begin{aligned} B-C &= \frac{1}{\sqrt{d}}\left[\frac{2A}{\kappa\varsigma}+(1-A)\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)\right] \\ &= \frac{1}{\sqrt{d}}\left[1+\frac{\beta+\delta}{\kappa\varsigma}+A\frac{(1-\beta)+(1-\delta)-\kappa\varsigma}{\kappa\varsigma}\right], \end{aligned}$$

so if $A=0$, then $B-C=\frac{1}{\sqrt{d}}\left[1+\frac{\beta+\delta}{\kappa\varsigma}\right]>1$, as we already established that $\sqrt{d}<1+\frac{\beta+\delta}{\kappa\varsigma}$. Hence, for all ϱ sufficiently close to 1, $B-C>1$. Therefore:

$$\begin{aligned} m_S &> A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \frac{1}{2^S}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^S + \frac{1}{2^S}\sum_{k=0}^{S-1}\binom{S}{k}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^k\sqrt{d}^{S-k} \\ &> A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \frac{1}{2^S}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^S + \frac{\sqrt{d}^S}{2^S}\sum_{k=0}^{S-1}\binom{S}{k} \\ &= A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S + \left(\frac{1}{2}+\frac{\beta+\delta}{2\kappa\varsigma}\right)^S + \frac{2^S-1}{2^S}\sqrt{d}^S. \end{aligned}$$

When $A=0$, this implies that $m_S>1$. Now, we are assuming that $1-\beta\delta>(1-\beta)(1-\delta)-\kappa\varsigma$, so $\kappa\varsigma+\beta+\delta>2\beta\delta$, and hence $\frac{1}{2}+\frac{\beta+\delta}{2\kappa\varsigma}>\frac{\beta\delta}{\kappa\varsigma}$ as $\kappa\varsigma>0$. Thus, the positive $\left(\frac{1}{2}+\frac{\beta+\delta}{2\kappa\varsigma}\right)^S$ term asymptotically dominates the negative $A\left(\frac{\beta\delta}{\kappa\varsigma}\right)^S$ term. Consequently, for all ϱ sufficiently close to 1, and all $S\geq 1$, $m_S>1$, as required. I.e., as long as $\varrho\in[0,1)$ is large enough, then the sub-matrix we started with will have determinant greater than one, no-matter how large it was. In this case, all principal minors of M are greater or equal to one, meaning that the M matrix is a "P-matrix" (Holden 2023), and moreover that no sufficiently small changes to the model could change this result.[6] (Being a P-matrix only requires positive principal minors, not ones greater than one.) Thus, with p_t^* exogenous, the solution is robustly unique conditional on the terminal conditions (bounded inflation, eventual escapes from both bounds).

Ruling out sunspot equilibria. As a final check of the performance of price level real rate rules, we examine whether they rule out persistent sunspot equilibria, following Subsection 4.3 of the main paper. We assume the model is given by equations (10) and (11), much as before, with the price level real rate rule introduced in this appendix. We assume that $p_t^* = \pi^*(t-1)$ and that $n + \pi^* > \epsilon > 0$, $\theta > 0$, $\kappa\varsigma > 0$ and $(1-\beta)(1-\delta) - \kappa\varsigma < 0$, again following Subsection 4.3. Suppose then that in period t for all $t \leq 0$, the economy was away from the ZLB, and was expected to stay there with probability one. Thus, by period 0, the impact of initial conditions must have dissipated, and so $p_0 = \breve{p}_0^* = p_0^*$.[7] Thus, $i_0 - r_0 = \mathbb{E}_0 p_1 - p_0 = \mathbb{E}_0 \breve{p}_1^* - p_0 = \breve{p}_1^* - \breve{p}_0^* = \pi^*$. However, in period 1, a "zero probability sunspot shock" hits, so that with probability one, for all $t \geq 1$, $0 = i_t = r_t + \pi_{t+1}$. (The expectation drops out of the Fisher equation as there is no other uncertainty.) Thus for $t \geq 1$, the Phillips curve and Euler equation imply that $\pi_t = \pi_Z$ and $x_t = x_Z$ where:

$$(1-\beta)(\pi_Z - \pi^*) = \kappa x_t, \qquad (1-\delta)x_Z = \varsigma(\pi_Z + n),$$

so,

$$\pi_Z - \pi^* = \frac{\kappa\varsigma(n+\pi^*)}{(1-\beta)(1-\delta) - \kappa\varsigma} < 0.$$

This is consistent with equilibrium if and only if the interest rate would be non-positive for $t \geq 1$ were it not for the ZLB. In period 1, this requires:

---

[6] As in Appendix K.9, this robustness holds for any fixed size M matrix. I.e., fix $T > 0$ (potentially extremely large) and suppose the bound ceases to apply more than T periods in the future. Then following a sufficiently small change to the model, there will be a unique solution that satisfies the bound for T periods, but which may violate it after T periods.

[7] Slightly more formally, we could suppose that the rule was introduced in period $-k$, and take the limit as $k \to \infty$, giving the same conclusion.

$$
\begin{aligned}
0 &\geq r_1 + \mathbb{E}_1 \breve{p}_2^* - \breve{p}_1^* + \theta(p_0^* + \pi_Z - \breve{p}_1^*) \\
&= \max\{0, \epsilon + \pi_Z - \pi^*\} + (\theta - 1)(\pi_Z - \pi^*).
\end{aligned}
$$

However, if $\theta < 1$, then $(\theta - 1)(\pi_Z - \pi^*) > 0$, so the condition cannot possibly hold. Thus, as long as the central bank does not respond too aggressively to the price level, there cannot be sunspot solutions of the kind previously described. Furthermore, it follows that as long as the economy is currently sufficiently close to the "good" steady-state, there is no way for the economy to ever jump to the ZLB. Thus, the price level real rate rule delivers robust uniqueness, even in the presence of the ZLB.

H.3 Perpetuity real rate rules

An even more robust solution to the problems caused by the ZLB is for the central bank to intervene in a market which does not have an equivalent to the ZLB. Perpetuities (also called "consols") are one such asset. For suppose that nominal interest rates were expected to be at i for all time. Then the price of a perpetuity would be $\frac{1}{i}$.[8] Thus, any finite, positive, perpetuity price is consistent with at least one path for future nominal interest rates. In other words, there is no upper or lower bound on the price of a perpetuity.

Note that the central bank does not strictly need the treasury to issue perpetuities in order to implement a perpetuity real rate rule. Since central banks in developed nations are generally believed to be extremely long-lived institutions, the central bank can issue perpetuities itself. As central banks can always print money to pay the coupon, central banks may be one of the only institutions that could be trusted to pay coupons for ever. Central banks may also decide to trust the perpetuities issued by some selected private banks, even if these will always carry some default risk. If the central bank views default as very unlikely in the short to medium term, then such default risk may not substantially distort pricing.

In the below, we will call standard perpetuities "nominal perpetuities". To implement a real rate rule on perpetuities, we will also need there to be a corresponding "real perpetuity" traded in the economy. In particular, we suppose

[8] This is correct under continuous time with a continuous flow of coupons, and approximately correct under discrete time, as we will see below.

that one unit of the period t nominal perpetuity bought at t returns \$1 at $t+1$, along with one unit of the period $t+1$ nominal perpetuity. On the other hand, one unit of the period t real perpetuity bought at t returns $\$\frac{P_{t+1}}{\Pi^{*t+1}}$ at $t+1$, along with one of the period $t+1$ real perpetuity, where P_{t+1} is the price level in period $t+1$ and $\Pi^* \geq 1$ is the target for the gross inflation rate. The nominal perpetuity trades at a price of $Q_{I,t}$ at t, whereas the real perpetuity trades at a price of $Q_{R,t}$ at t.

If we write Ξ_{t+1} for the real SDF between periods t and $t+1$, and $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ for gross inflation between these periods, then the price of these two perpetuities must satisfy:

$$Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}}[Q_{I,t+1} + 1], \qquad Q_{R,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}}\left[Q_{R,t+1} + \frac{P_{t+1}}{\Pi^{*t+1}}\right].$$

The real perpetuity price could be non-stationary due to the potential unit root in the logarithm of the price level, so it is helpful to define a detrended version. In particular, let:

$$\hat{Q}_{R,t} := Q_{R,t}\frac{\Pi^{*t}}{P_t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi^*}[\hat{Q}_{R,t+1} + 1].$$

Rewritten in this way, the analogy between the pricing of nominal and real perpetuities is clear. If $\Pi_t = \Pi^*$ for all t, then $Q_{I,t} = \hat{Q}_{R,t}$ for all t as well. If inflation and the SDF are stationary, then $\hat{Q}_{R,t}$ and $Q_{I,t}$ will admit a stationary solution.

We also assume that one period nominal bonds are traded in the economy, with gross return I_t. As in Subsection 7.1 of the main text, the pricing for these bonds must satisfy:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1.$$

We can now redo the argument of this subappendix's initial paragraph, slightly more formally. So, suppose that the gross nominal interest rate I_t is pegged at the constant level I (which may be inconsistent with the inflation target of Π^*). Then, the pricing equation for nominal perpetuities has a solution in which $Q_{I,t} = Q_I$ for all t, with $Q_I = I^{-1}[Q_I + 1]$, since $\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = I^{-1}$, for all t. Thus, $Q_I = \frac{1}{I-1}$. As $I \to 1$ (the ZLB), $Q_I \to \infty$, while as $I \to \infty$, $Q_I \to 0$. Thus, in line with our initial argument, any finite, positive, nominal perpetuity price is consistent with at least one possible path for nominal rates, no matter the dynamics of the real SDF. This ensures that the central bank can set the nominal perpetuity price to an arbitrary level, without any constraints. We do not need the real perpetuity price to be unbounded in this manner, as the central bank will not intervene in real perpetuity markets.

The reader might worry that a bound on nominal perpetuity prices could enter another way. Suppose that nominal perpetuity prices were known at least one period in advance (e.g., because there is no uncertainty), and that money is available to trade. Then it would be the case that $Q_{I,t+1} + 1 \geq Q_{I,t}$, else nominal perpetuities would have return strictly dominated by that of cash. This inequality is an immediate consequence of $I_t \geq 1$ though, when $Q_{I,t+1}$ is known at t. $I_t \geq 1$ implies $\frac{Q_{I,t}}{I_t} \leq Q_{I,t}$, so:

$$Q_{I,t} \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = \frac{Q_{I,t}}{I_t} \leq Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [Q_{I,t+1} + 1],$$

which implies $Q_{I,t+1} + 1 \geq Q_{I,t}$ if $Q_{I,t+1}$ is known at t. Thus, the bound on one period nominal rates is all that really matters, and we have already showed that this bound does not imply a bound on $Q_{I,t}$. Intuitively, $Q_{I,t+1} + 1 \geq Q_{I,t}$ is not a constraint on $Q_{I,t}$ as $Q_{I,t+1}$ is endogenous.

We can now introduce our perpetuity real rate rule. We suppose that the central bank intervenes in nominal perpetuity markets to ensure:

$$Q_{I,t} = \hat{Q}_{R,t} \left(\frac{\Pi_t}{\Pi^*} \right)^{-\psi},$$

for some exponent $\psi \in \mathbb{R}$. While $\psi > 0$ may seem natural (so that high inflation results in low bond prices and thus high interest rates), we do not impose this.

We analyse the resulting dynamics via log-linearizing around the steady-state with inflation at Π^*.[9] In particular, suppose that:

$$Q_{I,t} = Q \exp q_{I,t}, \qquad \hat{Q}_{R,t} = Q \exp q_{R,t},$$
$$\Xi_t = \Xi \exp \xi_t, \qquad \Pi_t = \Pi^* \exp \pi_t,$$

where $Q := \frac{1}{I^* - 1}$, with $I^* := \frac{\Pi^*}{\Xi}$. We assume $\Xi < 1$, so $I^* > 1$. Then to a first order approximation around $q_{I,t} = q_{R,t} = \xi_t = \pi_t = 0$:

$$q_{I,t} = \mathbb{E}_t \left[\xi_{t+1} - \pi_{t+1} + \frac{\Xi}{\Pi^*} q_{I,t+1} \right], \qquad q_{R,t} = \mathbb{E}_t \left[\xi_{t+1} + \frac{\Xi}{\Pi^*} q_{R,t+1} \right],$$
$$q_{I,t} = q_{R,t} - \psi \pi_t.$$

[9] While it would ideally be better to examine these determinacy questions in a fully non-linear model, this is not tractable. We take comfort from the fact that even Cochrane (2011) primarily relies on linearized models.

Thus:

$$\psi\pi_t = q_{R,t} - q_{I,t} = \mathbb{E}_t\left[\pi_{t+1} + \frac{\Xi}{\Pi^*}(q_{R,t+1} - q_{I,t+1})\right] = \mathbb{E}_t\left[\pi_{t+1} + \frac{\Xi}{\Pi^*}\psi\pi_{t+1}\right].$$

Hence, if we define $\phi := \psi\left[1 + \frac{\Xi}{\Pi^*}\psi\right]^{-1}$, we then have that $\phi\pi_t = \mathbb{E}_t\pi_{t+1}$, just as when one period bonds are used. With $\phi > 1$, this has the unique stationary solution $\pi_t = 0$ (so $\Pi_t = \Pi^*$), as usual. The crucial difference is that with the perpetuity real rate rule, this is achieved without violating the ZLB.

As a final observation, note that our definition of ϕ implies that $\psi = -\phi\left[\frac{\Xi}{\Pi^*}\phi - 1\right]^{-1}$, so, for sufficiently large ϕ ($\phi > I^* = \frac{\Pi^*}{\Xi}$) $\psi < -\frac{\Pi^*}{\Xi} < 0$. Thus, under a perpetuity real rate rule with sufficiently large ϕ, the central bank will raise nominal perpetuity prices in response to high inflation. This sign becomes more intuitive once money flows are considered. While if the central bank buys perpetuities, they are raising the money supply in the period of purchase, in every subsequent period they are reducing the money supply, as the private sector must pay coupons back to the central bank. Given the forward-looking nature of inflation determination, it is this long-run reduction which is crucial.

Appendix I The empirical performance of the Fisher equation

For real rate rules to work, the Fisher equation must hold at least approximately with whatever assets the central bank considers using. There may be time-varying wedges in the Fisher equation from liquidity and risk premia, and there could even be a non-unit coefficient on expected inflation,[10] but there must be at least some relationship between $i_t - r_t$ and expected inflation.

Since our main empirical exercise will use five-year TIPS and treasuries, we will be most interested in whether the Fisher equation holds for these assets. We will perform our tests using monthly data on five-year breakeven inflation rates constructed from five-year treasuries and five-year TIPS. Breakeven inflation rates give us a measure of $i_t - r_t$ for these assets.[11]

[10] If $i_t - r_t = \kappa\mathbb{E}_t\pi_{t+1}$ and the central bank uses the simple rule $i_t - r_t = \phi\pi_t$, then $\mathbb{E}_t\pi_{t+1} = \frac{\phi}{\kappa}\pi_t$, so $\left|\frac{\phi}{\kappa}\right| > 1$ is necessary and sufficient for determinacy. If $\kappa \in (0,1)$, then $\phi > 1$ is sufficient.

[11] We obtain breakeven inflation from Federal Reserve Bank of St. Louis (2024a), and converted to a continuously compounded rate.

Unsurprisingly, there is much prior work testing the Fisher equation. Balfoussia & Wickens (2006) find evidence in favour of the Fisher equation when real rates are inferred from ex-post real returns. Zeng (2013) finds that breakeven inflation rates tend to underestimate inflation expectations, due to liquidity premia. However, static wedges in the Fisher equation are removed by differencing, and so pose no challenge to the use of real rate rules. Abrahams et al. (2016) find that beyond the five-year horizon, breakeven rates are mostly driven by changes in risk premia, which justifies our focus on five-year breakeven rates. Bennett & Owyang (2023) survey the literature on the substantial role of inflation and liquidity premia in driving breakeven inflation rates. They go on to test the forecasting performance of five-year TIPS breakeven rates, and find them to be more accurate than VARs or breakeven rates from inflation swaps. They also find that TIPS breakeven rates are unbiased, unlike breakeven rates from inflation swaps. Using UK data, Scholtes (2002) finds that UK two-year breakeven rates outperform professional forecasters in forecasting RPI inflation.

In the rest of this appendix, we perform two additional tests of the Fisher equation. We first check that professional forecasts predict breakeven rates, and then we check that breakeven rates predict realised inflation. Throughout Appendix I we convert all rates into continuously compounded ones, which are 100 times the difference in logarithms of the price levels.

I.1 Do professional forecasts of inflation predict breakeven rates?

We first examine the association between breakeven inflation and professional forecasts from the Survey of Professional Forecasters (SPF). SPF forecasts are produced quarterly. We use the median across respondents given that surveys tend to have fat tails.[12]

Each survey contains a prediction for five-year CPI inflation over the period starting from quarter four of the year before the survey year, where quarterly CPI levels are averages of monthly CPI levels. For simplicity, we approximate this by assuming instead that logarithms of quarterly levels are averages of logarithms of monthly levels. This approximation implies that once converted into a

[12] Taken from Federal Reserve Bank of Philadelphia (2024).

continuously compounded rate, the reported quantity is $\frac{1}{180}$ times the annualized continuously compounded (ACC) inflation rate in November the year before, plus $\frac{1}{90}$ times the ACC inflation rate in December the year before, plus $\frac{1}{60}$ times the ACC inflation rate in January of the survey year, ..., plus $\frac{1}{60}$ times the ACC inflation rate in October of the year four years after the survey year, plus $\frac{1}{90}$ times the ACC inflation rate in November of the year four years after the survey year, plus $\frac{1}{180}$ times the ACC inflation rate in December of the year four years after the survey year.

Surveys have a deadline in the second week of month two of the quarter, which is usually before the release of CPI inflation for the first month of that quarter. For simplicity, we assume this is always the case, as in any case, many surveyed will submit their answers before the deadline. Thus, we treat the final month of the previous quarter as the last month observed by survey participants. Given this, it is natural to compare forecasts to breakeven rates from the first month of the survey quarter, as these are based on bonds priced with the same information set. Given the indexation lag in breakeven inflation, this means that in quarter one, breakeven rates cover inflation from October of the year before; in quarter two, breakeven rates cover inflation from January; in quarter three, breakeven rates cover inflation from April; and, in quarter four, breakeven rates cover inflation from July. However, SPF forecasts are always the average of three forecasts, one from November of the year before, one from December of the year before, and one from January of the current year. We construct modified breakeven and SPF forecasts which start from the period covered by breakeven forecasts in quarters two to four, but which matches the SPF forecast period in quarter one.

In quarter one of each year, we need to construct three modified breakeven forecasts starting from one month later (November, not October), two months later (December) and three months later (January). We create the first by subtracting $\frac{1}{60}$ times realized ACC inflation rate for October from the forecast, and then adding back $\frac{1}{60}$ times the five-year, five-year forward breakeven rate derived from five and ten-year TIPS and treasuries.[13] This treats the five-year, five-year

[13] Taken from Federal Reserve Bank of St. Louis (2024b), and converted to a continuously compounded rate.

forward breakeven rate as a measure of market long-run inflation expectations. For the second, we subtract $\frac{1}{60}$ times the sum of the ACC rates of October and November, and add back $\frac{1}{30}$ times the five-year, five-year forward breakeven rate. For the third, we subtract $\frac{1}{60}$ times the sum of the ACC rates of October, November and December, and add back $\frac{1}{20}$ times the five-year, five-year forward breakeven rate. The final quarter one modified breakeven inflation forecast is the average of these three.

In the other quarters of the year, we need to construct a modified survey forecast starting from either two months later (in quarter two), five months later (in quarter three) or eight months later (in quarter four). We do this by subtracting realized ACC inflation rates for the future months (from the perspective of the forecast), multiplied by the appropriate weights $(\frac{1}{180}, \frac{1}{90}, \frac{1}{60}, \frac{1}{60}, \ldots)$, and then adding back the forecast of five-year, five-year forward inflation, multiplied by the sum of the same weights. This treats the five-year, five-year forward SPF forecast as a measure of SPF forecasters' long-run inflation expectations.

Having done all this, we then have a consistent set of breakeven and SPF inflation forecasts. We plot the relationship between the two over the full period of available data (2005 Q3 to 2024 Q1) in Figure 1.[14] The figure also shows the result of a regression of five-year breakeven inflation on five-year SPF inflation expectations and a constant. The estimated slope is 0.90, with a heteroskedasticity and autocorrelation (HAC) robust standard error of 0.17. This gives a p-value for the null hypothesis of zero slope below 10^{-4}, and a p-value for the null hypothesis of a unit slope of 0.53. Thus, we cannot reject the null that breakeven inflation responds one-for-one to movements in inflation expectations, in line with the Fisher equation. This provides strong support for the Fisher equation, as long as we allow for it to contain a stochastic wedge.

[14] This figure and the accompanying regression can be generated by running the MATLAB script "Main.m" provided in this paper's replication materials.

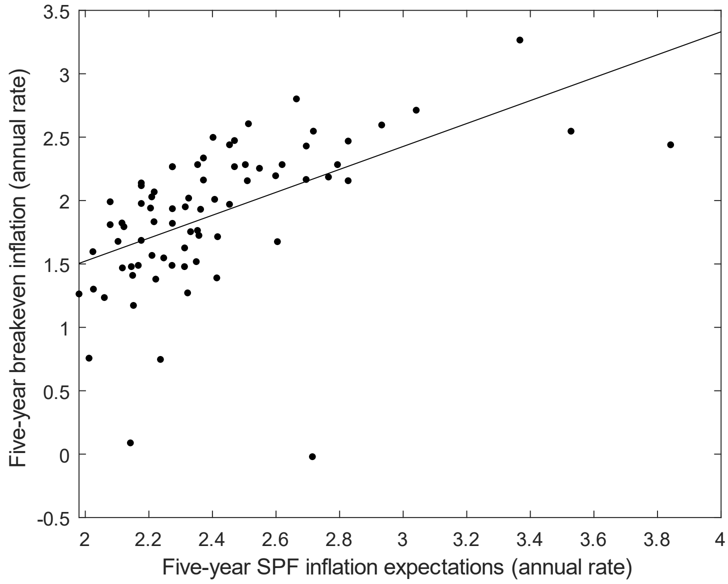

Figure 1: The relationship between five-year SPF inflation expectations and five-year breakeven inflation rates, over matched horizons.

I.2 Do breakeven rates forecast inflation?

We now examine whether five-year breakeven rates contain information useful for forecasting inflation. For each month's observation of breakeven inflation, we construct the five-year average of realised CPI inflation over the same horizon as used by the TIPS, taking account of the three-month indexation lag.[15] Under rational expectations:

$$\frac{1}{60}\sum_{k=0}^{59}\pi_{t+k} = \mathbb{E}_t\frac{1}{60}\sum_{k=0}^{59}\pi_{t+k} + \text{an unexpected shock},$$

(where t is monthly), so if the Fisher equation holds, then realized inflation $(\frac{1}{60}\sum_{k=0}^{59}\pi_{t+k})$ equals breakeven inflation plus an unexpected shock.

Regressing average realized inflation on breakeven inflation and a constant gives an estimated slope of 0.25, but the standard errors are unreliable as the error is near $I(1)$. However, since our preferred "practical" real rate rule specification is in terms of changes in $i_t - r_t$, it makes more sense to instead examine whether *changes* in breakeven inflation forecast *changes* in average realized inflation. Differencing ensures the errors are stationary and removes the slow-moving component of the Fisher equation wedge. Regressing changes in realized inflation on changes in breakeven inflation (without a constant) gives an estimated slope of

[15] Our sample for breakeven inflation is from January 2003 to June 2019.

0.09, with a heteroskedasticity and auto-correlation robust standard error of 0.02 (and corresponding P-value below 10^{-4}).[16] Thus, breakeven inflation contains statistically significant information about future inflation. We plot this relationship in Figure 2.[17] Note that since the coefficient is likely to be heavily biased towards zero due to the noise in breakeven rates coming from fluctuating risk premia, we cannot infer from this that the slope is necessarily different than one.

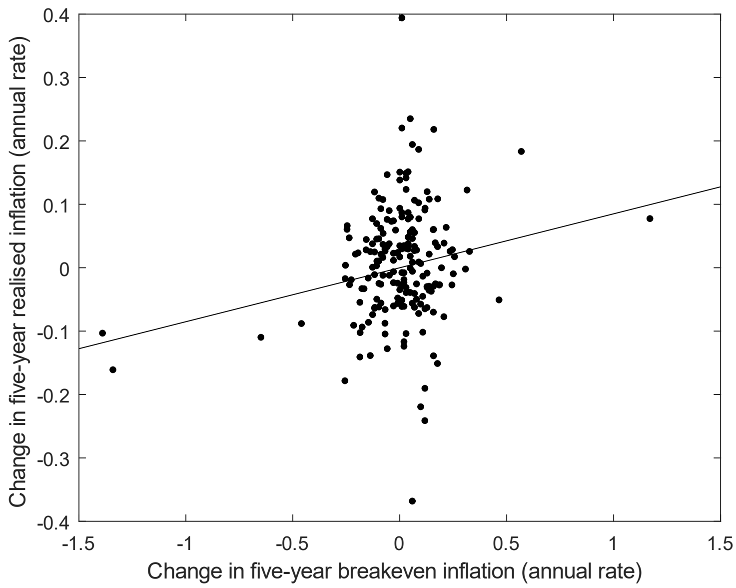

Figure 2: The relationship between changes in five-year breakeven inflation rates, and five-year realised inflation over the same horizon.

Appendix J Details of the empirical exercise

J.1 Background on the Summary of Economic Projections (SEP)

The United States Federal Open Market Committee releases a "Summary of Economic Projections" (SEP) approximately once every three months. This contains statistics summarising the projections of the seven Federal Reserve board members and the twelve Federal Reserve bank presidents. Crucially, these projections are conditional on the Fed following what the individual believes to be "appropriate monetary policy":[18]

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[16] This result and the previous one may be generated by running the MATLAB script "Main.m" provided in this paper's replication materials.

[17] This figure can also be generated by running the MATLAB script "Main.m" provided in this paper's replication materials.

[18] Federal Open Market Committee (2024).

> *"Each participant's projections [are] based on information available at the time of the meeting, together with her or his assessment of appropriate monetary policy—including a path for the federal funds rate and its longer-run value—and assumptions about other factors likely to affect economic outcomes. The longer-run projections represent each participant's assessment of the value to which each variable would be expected to converge, over time, under appropriate monetary policy and in the absence of further shocks to the economy. 'Appropriate monetary policy' is defined as the future path of policy that each participant deems most likely to foster outcomes for economic activity and inflation that best satisfy his or her individual interpretation of the statutory mandate to promote maximum employment and price stability."*

We will use the projections for the PCEPI inflation rate, both for the next few years, and for the long-run. Since these are projections for what inflation ought to be if monetary policy is set optimally, we take them as capturing π_t^* from our model of a time-varying short-run inflation target.[19]

While recent releases of the SEP contain information on the median forecast across participants, this is not available for the full sample (from November 2007). Instead, we have to rely on the mid-point of the central tendency. This is the average of the fourth largest forecast, and the fourth smallest forecast. (Recall that at most 19 individuals give projections each round.)

To give an indication of the reliability of the mid-point of the central tendency as a measure of a distribution's location, Table 1 gives the raw moments (from a sample of size 10^8) of the absolute value of the mean, median and central-tendency mid-point of samples of size 19 from a standard T-distribution with degrees of freedom parameter 5, and Table 2 repeats this for draws from a standard normal distribution.[20] While the central tendency mid-point is less efficient than either the mean or the median in the T-distribution case, the difference is not

[19] See Kocherlakota (2023) for more background on the Summary of Economic Projections.

[20] These tables may be generated by running the MATLAB script "Main.m" provided in this paper's replication materials. For the mean of a normal distribution, closed form analytical moments are available, which agree perfectly with the numbers given in Table 2.

massive (below 30% for all moments). In the Gaussian case, the mid-point of central tendency is more efficient than the median for all moments, but its efficiency loss relative to the mean is larger (about 61% for the fourth moment). Overall, these results suggest that the mid-point of the central tendency is a reasonable measure of the location of the centre of the distribution of projections.

Raw moment	Mean	Median	Central tendency mid-point
1	0.234	0.240	0.250
2	0.088	0.092	0.100
3	0.043	0.045	0.052
4	0.026	0.026	0.033

Table 1: Raw moments of the absolute value of the mean, median and central-tendency mid-point of samples of size 19 from a standard T-distribution with degrees of freedom parameter 5.

Raw moment	Mean	Median	Central tendency mid-point
1	0.183	0.227	0.206
2	0.053	0.081	0.067
3	0.019	0.037	0.027
4	0.008	0.020	0.013

Table 2: Raw moments of the absolute value of the mean, median and central-tendency mid-point of samples of size 19 from a standard normal distribution.

J.2 From PCEPI inflation expectations to CPI inflation expectations

In the Summary of Economic Projections, participants give forecasts for PCEPI inflation, in line with the Fed's target being stated in terms of PCEPI inflation. However, the pay-off of TIPS is a function of CPI inflation. Thus, we need to convert projections from PCEPI inflation to CPI inflation.

In order to avoid contaminating the projections with information not available at the time, we work with the first released estimates of the seasonally adjusted values of monthly, continuously compounded, CPI inflation and PCEPI inflation,[21] unless these first released estimates are updated within one month of

[21] Obtained from Federal Reserve Bank of St. Louis (2024c) (based on underlying data from U.S. Bureau of Labor Statistics (2024)) and Federal Reserve Bank of St. Louis (2024d) (based on underlying data from U.S.

the end of the month for which inflation is being measured, in which case we use the updated estimates. Throughout Appendix J, continuously compounded inflation rates are 100 times the difference in logarithms of the price levels. The estimate of CPI inflation for a month is always released less than a month after the end of the month in question in our sample (median and mean delay in our sample: 14 days; maximum delay: 29 days), while for PCEPI inflation it can sometimes take over a month, but less than two (median and mean delay in our sample: 28 days; maximum delay: 59 days). For simplicity, we treat both series as having a one-month release lag though.

Let π_t^{CPI} and π_t^{PCE} be the estimates of continuously compounded CPI and PCEPI inflation for month t, (released in month $t+1$). We estimate the following time-varying linear regression on data from January 2002 to February 2024:[22]

$$\pi_t^{\text{CPI}} = \alpha_t + \beta_t \pi_t^{\text{PCE}} + \sigma_\varepsilon \varepsilon_t, \qquad \varepsilon_t \sim \text{N}(0,1),$$
$$\alpha_t = \alpha_{t-1} + \sigma_{\alpha\alpha}\nu_{\alpha,t} + \sigma_{\alpha\beta}\nu_{\beta,t}, \qquad \alpha_0 \sim \text{N}(0,1), \qquad \nu_{\alpha,t} \sim \text{N}(0,1),$$
$$\beta_t = \beta_{t-1} + \sigma_{\beta\beta}\nu_{\beta,t}, \qquad \beta_0 \sim \text{N}(0,36), \qquad \nu_{\beta,t} \sim \text{N}(0,1).$$

We take $\beta_0 \sim \text{N}(0,36)$ as we expect the mean of π_t^{PCE} to be about two (percent annual inflation target) divided by twelve (months in a year), so this choice ensures $\beta_0 \pi_t^{\text{PCE}} | \pi_t^{\text{PCE}}$ is roughly distributed as $\text{N}(0,1)$, just like α_0.

We estimate (sandwich (robust/QMLE) standard errors in brackets) $\sigma_{\alpha\alpha} \approx 0.0024$ (0.0028), $\sigma_{\alpha\beta} \approx -0.0050$ (0.0024), $\sigma_{\beta\beta} \approx 0.0171$ (0.0081), $\sigma_\varepsilon \approx 0.0789$ (0.0045), via maximum likelihood. We plot the estimated (smoothed) paths of α_t and β_t in Figure 3 and Figure 4.[23] We also plot the implied correlation between PCEPI and CPI inflation in Figure 5. We will convert forecasts made in month t from PCEPI inflation to CPI inflation using the smoothed estimates of α_{t-1} and β_{t-1}. For simplicity, we ignore the uncertainty associated with these estimates in all of the following exercises.

Bureau of Economic Analysis (2024)) respectively.

[22] We start in 2002 rather than say 2007 (when the Summary of Economic Projections data starts) in order to have a sufficient run-in for the impact of initial conditions to dissipate.

[23] These estimates and figures may be produced by running the MATLAB script "Main.m" provided in this paper's replication materials.

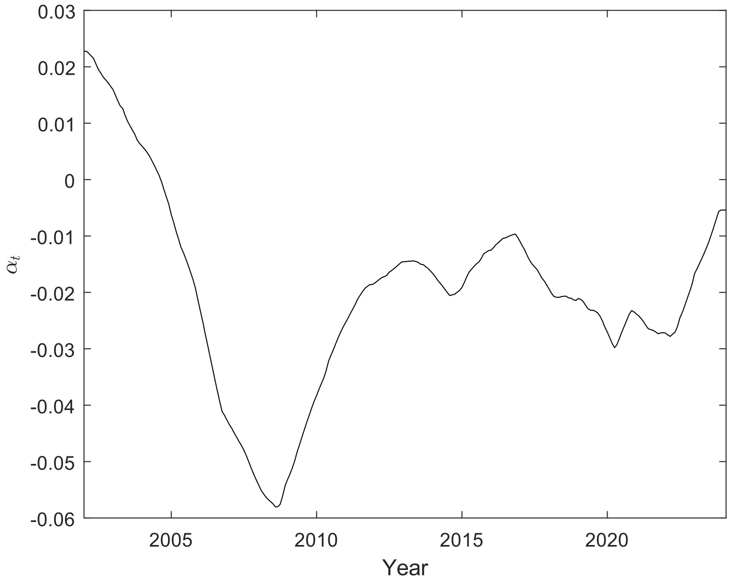

Figure 3: Estimated (smoothed) value of α_t.

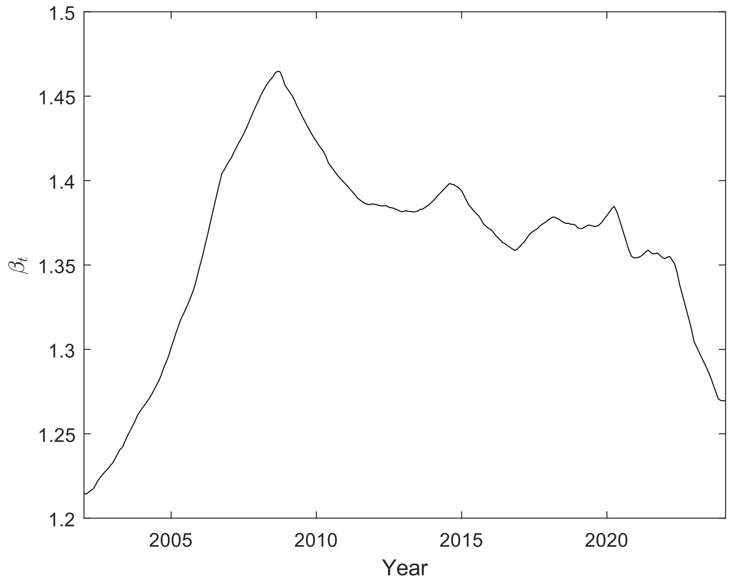

Figure 4: Estimated (smoothed) value of β_t.

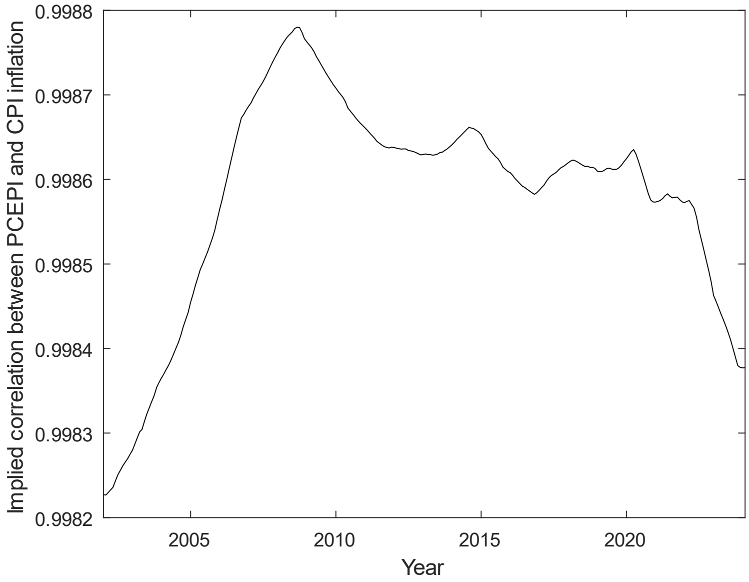

Figure 5: Implied correlation between PCE and CPI inflation.

J.3 A simple annual exercise

The Summary of Economic Projections contains forecasts for PCEPI inflation over particular calendar years.[24,25] Using our time-varying mapping PCEPI to CPI inflation map, we can convert these to CPI forecasts. Averaging all (CPI) forecasts made in a year gives us an annual forecast "made" in that year. Thus, if we work with an annual frequency model, then we have the relevant forecasts to estimate the monetary rule, without doing anything very sophisticated. It is simple enough that the entire exercise could be performed using spreadsheets.

In particular, we want to estimate θ in the following smoothed real rate rule, where now t is measured in years, and where we have substituted in the Fisher equation for five year bonds to remove $i_t - r_t$ and $i_{t-1} - r_{t-1}$:

$$\mathbb{E}_t \frac{1}{5}\sum_{k=1}^{5}(\pi_{t+k} - \pi^*_{t+k}) - \mathbb{E}_{t-1}\frac{1}{5}\sum_{k=1}^{5}(\pi_{t-1+k} - \pi^*_{t-1+k}) = \theta(\pi_t - \pi^*_t).$$

We do not worry about observation or indexation lags, as these are small relative to the length of a time period (a year).

The only data missing is forecasts at longer horizons, as each SEP release only contains forecasts for three or four years, including the current year, but we need

[24] Technically, they are Q4 to Q4 forecasts.

[25] The SEP inflation forecast data is taken from Federal Reserve Bank of St. Louis (2024e) (based on underlying data from U.S. Federal Open Market Committee and Federal Reserve Bank of St. Louis (2024a)).

forecasts for six years, including the current one. We estimate an AR(1) model by regressing:

- SEP annual (CPI converted from PCEPI) inflation forecasts at the longest horizon available, minus the long-run SEP (CPI converted from PCEPI) inflation forecast,[26]

on:

- SEP annual (CPI converted from PCEPI) inflation forecasts at the previous horizon, minus the long-run SEP (CPI converted from PCEPI) inflation forecast,

using the full SEP data set of vintages from November 2007 to March 2024. This gives an estimate of the perceived persistence of desired inflation according to the SEP. We obtain a persistence of 0.55 (HAC standard error: 0.10).[27] We then forecast projections beyond the available horizon using this estimated AR(1) model, and the relevant SEP long-run inflation forecast. Averaging over all vintages in a year then gives us $\mathbb{E}_t \frac{1}{5} \sum_{k=1}^{5} \pi^*_{t+k}$ and π^*_t.

We just need data on $\mathbb{E}_t \frac{1}{5} \sum_{k=1}^{5} \pi_{t+k}$ and π_t. For the former, we use breakeven inflation constructed from five-year treasuries and five-year TIPS, as discussed in Appendix I.[28] Since we need an end of year value, as all terms in $\mathbb{E}_t \frac{1}{5} \sum_{k=1}^{5} \pi_{t+k}$ are future dated, we use the average of the December and following January observations. Likewise, we use the average of the December and following January observations of annual CPI inflation to obtain π_t.[29]

Figure 6 plots the data, and an estimated slope line with zero intercept. The first observation is 2009, as there is a lag in the model (consuming the 2008 data point), and there is too little SEP data in 2007 to use that year. The final observation is 2023. There is a clear positive association, as would be expected were the Fed close to following a real rate rule. The R^2 value is 0.46 meaning that this simple linear model is capturing around half of the variance in the data. θ is estimated at

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[26] The long-run SEP inflation forecasts are taken from U.S. Federal Open Market Committee and Federal Reserve Bank of St. Louis (2024b) and are extrapolated backwards with their first observation.

[27] This entire exercise is performed in the MATLAB script "Main.m".

[28] We obtain breakeven inflation from Federal Reserve Bank of St. Louis (2024a).

[29] We take CPI data from U.S. Bureau of Labor Statistics (2024).

0.513, with a heteroskedasticity and autocorrelation robust p-value of 0.006. Of course, endogeneity is a concern here. We address this in the quarterly estimates that follow.

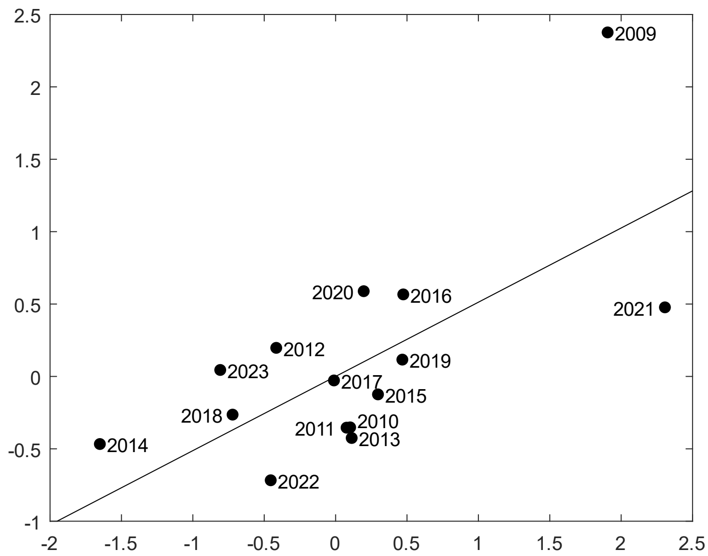

Figure 6: Annual real rate rule data and estimated slope.

J.4 Inferring monthly or quarterly short-run inflation targets from the SEP

In averaging to annual, we threw away a lot of data, and thus ended up with less precise estimates. The difficulty with using quarterly or monthly data though, is that we only observe forecasts for inflation over particular calendar years, not for specific quarters. Furthermore, the month in which the forecasts are made changes over the sample. Thus, we need a model to infer consistent monthly or quarterly observations of π_t^* from what we do observe. Given the changing observation months, it makes sense to work with a monthly model.

Writing L for the lag operator and I for the identity operator, and with t now being measured in months, we assume that:

$$\pi_{\infty,t}^* = (I - 0.9999L)^{-1}\sigma_\infty \varepsilon_{\infty,t},$$

$$\pi_{1,t}^* = (I - \rho_{1*}L)^{-1}(I + \psi_{1*}L)\sigma_{1*}\varepsilon_{1*,t},$$

$$\pi_{1,t} = (I - \rho_1 L)^{-1}(I + \psi_1 L)\sigma_1 \varepsilon_{1,t},$$

$$\pi_{2,t}^* = \sum_{k\in\{0,12,24,36\}} \frac{1}{1+k}\exp\left(\frac{k}{1+k}\right)\left(I - \exp\left(-\frac{1}{1+k}\right)L\right)^{-2}\sigma_2\varepsilon_{2,k,t},$$

$$\pi_t^* = \pi_{\infty,t}^* + \pi_{1,t}^* + \pi_{2,t}^*,$$
$$\pi_t = \pi_t^* + \pi_{1,t},$$

where $\varepsilon_{\infty,t}, \varepsilon_{1*,t}, \varepsilon_{2,0,t}, \varepsilon_{2,12,t}, \varepsilon_{2,24,t}, \varepsilon_{2,36,t}, \varepsilon_{1,t} \sim \mathrm{N}(0,1)$.

We unpick this term by term. $\pi_{\infty,t}^*$ gives a near unit root AR(1) component to π_t^* (and hence π_t) that will capture the changing long-run inflation target. $\pi_{1,t}^*$ gives an ARMA(1,1) component to π_t^*, and $\pi_{1,t}$ gives an ARMA(1,1) component to π_t. These are reasonable as ARMA(1,1) models tend to perform well for forecasting inflation, and it is not obvious a priori whether this is best captured by ARMA(1,1) fluctuations in π_t or π_t^*.

Finally, for any k:

$$\frac{1}{1+k}\exp\left(\frac{k}{1+k}\right)\left(I - \exp\left(-\frac{1}{1+k}\right)L\right)^{-2}\sigma_2\varepsilon_{2,k,t}$$

gives a repeated root AR(2) component with the following key properties. Firstly, the IRF to a unit shock to $\varepsilon_{2,k,t}$ in period 0 peaks in period k (the IRF is hump-shaped for $k > 0$). Secondly, the peak value of this IRF is σ_2, which is common across k to avoid over-parameterization. In particular, this IRF is given by:

$$\sigma_2\frac{1+t}{1+k}\exp\left(\frac{k-t}{1+k}\right).$$

This has a broadly similar shape to the Nelson & Siegel (1987) curvature factor, used for modelling inflation expectations by Aruoba (2020).[30] In our specific context, our approach has the advantage of enabling us to use the information in realised inflation, which helps make-up for the sparsity of the SEP data set.

We include these repeated root AR(2) terms for $k \in \{0,12,24,36\}$ as the SEP contains inflation forecasts at horizons of about zero, one, two and possibly three years. The $k = 0$ term will capture movements in the zero-year horizon projections, the $k = 12$ term will capture movements in the one-year horizon projections, and so on. Thus, we will not need to include any measurement error terms.

We estimate the model using monthly data from the same sources as in the previous subappendices, with data from January 2007 to March 2024.[31] The first

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[30] The slope and curvature factors are $O(t^{-1})$ as $t \to \infty$, which implies greater persistence than any stationary finite-order ARMA process. We instead capture a near permanent component with the $(I - 0.9999L)^{-1}\sigma_\infty\varepsilon_{\infty,t}$ term.

[31] The code for performing the estimation is contained in the MATLAB script "Main.m" in this paper's

SEP release is in November 2007, and the last is in March of 2024. The initial months of 2007 allow the smoother to infer something about the higher frequency state variables, before the SEP data starts.

In line with the SEP, we construct Q4 on Q4 annual inflation measures as December observations of $\frac{1}{3}\sum_{l=0}^{2}\sum_{k=0}^{11}\pi_{t-l-k}$. Expectations of such measures are constructed by iterating on the state space model's transition matrix. We include every available SEP inflation forecast, with each linked to the month in which it was released. Our observed measure of CPI inflation is the real time one detailed in J.2. We take the observed SEP long-run inflation forecasts as being observations of the $\pi^*_{\infty,t}$ term. (Since this series is constant almost everywhere, whenever two adjacent observations are the same, we interpolate to monthly with that value.) We assume that the initial state is drawn from the model's stationary distribution, but with the level modified so that any state that contains $\pi^*_{\infty,t}$ has mean two (percent annual inflation target) over twelve (months in a year). In order to avoid favouring $\pi^*_{1,t}$ over $\pi_{1,t}$, we start the optimization with identical parameters for these two ARMA(1,1) processes, taken from an initial estimate of an ARMA(1,1) on CPI.

We estimate (sandwich (robust/QMLE) standard errors in brackets) $\rho_1 \approx -0.03$ (0.14), $\psi_1 \approx 0.26$ (0.03), $\sigma_1 \approx 0.12$ (0.03), $\rho_{1*} \approx 0.64$ (0.05), $\psi_{1*} \approx 0.03$ (0.03), $\sigma_{1*} \approx 0.19$ (0.03), $\sigma_\infty \approx 0.0010$ (0.0004), $\sigma_2 \approx 0.0103$ (0.0010), via maximum likelihood. Thus, $\pi_{1,t}$ is essentially an MA(1) process, while $\pi^*_{1,t}$ is essentially an AR(1) process. This means that π_t does not deviate persistently from π^*_t, so the projections are tracking inflation well. It also means that movements in π^*_t explains much of the variance in π_t except at the highest frequencies. This is also clear from Figure 7 which plots the smoothed estimates of π^*_t and π_t, aggregated to quarterly frequency.

replication materials.

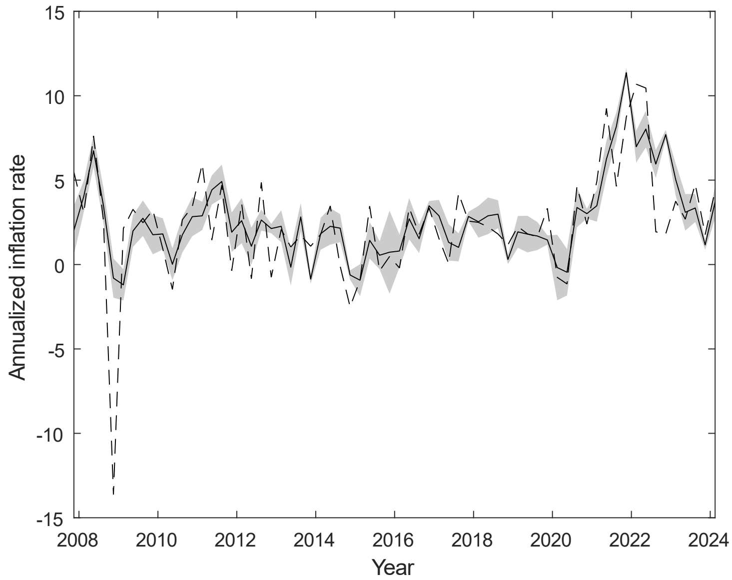

Figure 7: Estimated (smoothed) value of π_t^* (solid line, 90% confidence band in grey) and π_t (dashed line). Quarterly aggregates. Annualized continuously compounded rates.

We use quarterly frequency aggregates here and in our estimation exercise for two reasons. Firstly, since most macroeconomic models are calibrated or estimated on quarterly frequency data. Secondly, because the SEP is only released about once every three months. Thus, the monthly estimates of π_t^* do not really contain any more information than quarterly aggregates of this series. In effect, the additional variation in the monthly series is likely to be pure measurement error. By aggregating to quarterly, we reduce this measurement error, lessening the impact of "errors in variables" bias.

In our estimation exercise, for simplicity we ignore the uncertainty associated with these quarterly estimates of π_t^*. We drop the observations before the first SEP release in 2007 Q4 as these are unlikely to be uninformative, and we drop a further year of observations as the multi-year horizons of the SEP forecasts mean that early inferences about π_t^* will be less reliable than later ones. Thus, the first observation we use will be 2008 Q4 (except where a lag enters in which case we also use the 2008 Q3 observation for the lag).

Two data points in Figure 7 warrant further discussion. The implied Q3 and Q4 2021 values of π_t^* are above the realised values of π_t. This may be surprising! But this does come directly from the underlying SEP data. Mapped to CPI, the December 2021 SEP forecast for inflation over 2021 was 6.93%. Realised CPI inflation for 2021 came in at 6.62% (calculated with the formula $\frac{1}{3}\sum_{l=0}^{2}\sum_{k=0}^{11}\pi_{t-l-k}$ mentioned previously). So, realised CPI inflation for 2021 was below the Fed's SEP "target". Given that the SEP was below realised inflation for the first two quarters of 2021, the model needs a fairly sizeable overshoot in the final two quarters to hit the December 2021 SEP forecast. (In March and June of 2021, the SEP forecast for 2021 inflation (mapped to CPI) were 2.83% and 4.17% respectively, while Q1 and Q2 CPI inflation were 4.91% and 9.24%, annualized, respectively.)

The only remaining question is why the 2021 Q4 SEP forecast should be so high. Perhaps policy makers believe that by the time of the Q4 SEP, there is nothing they can do to influence that year's inflation. (But note that the December 2021 SEP release was prepared for the December 2021 FOMC meeting, and that the estimates of Miranda-Agrippino & Ricco (2021) imply monetary shocks have a same month impact on CPI.) Perhaps our estimated mapping from PCEPI to CPI is not accurate enough. Or perhaps it reflects a desire to make up for past low inflation during the ZLB period, given the Fed's average inflation targeting. Or to follow through on past "lower for longer" commitments. Perhaps it reflects a belief in the costliness of unanticipated interest rises. Or a desire to shrink the output gap. Given the Fed did not begin raising rates until March 2022, it does not seem outrageous that the model views monetary policy as dovish in 2021.

J.5 Estimating a real rate rule on quarterly data

Armed with data on π_t^*, we can now estimate the real rate rule of Subsection 5.2 form the main text, over the period 2008 Q4 to 2023 Q2. (The final date is the latest available observation of the Känzig (2021; 2024) oil price news shock we use as an instrument, as documented in the main text.) We work with five-year US treasuries and TIPS, so $T = 20$. Since annual yields on five-year US treasuries never dropped below 0.19% over our sample,[32] we ignore the ZLB. As suggested in the main text for the US, we take $L = 1$ (i.e., three months). We take $S = 0$, since the true CPI release delay of below one month is less than half of the length of a

[32] Data from Board of Governors of the Federal Reserve System (US) (2024a). The minimum is found by the MATLAB script "Main.m" in this paper's replication materials.

period (three months) and so for simplicity we write (e.g.) ν_t rather than $\nu_{t|t}$. Thus, we wish to estimate:

$$y_t := \mathbb{E}_t \frac{1}{T} \sum_{k=0}^{T-1} (\pi_{t+k} - \pi_{t+k}^*) - \mathbb{E}_{t-1} \frac{1}{T} \sum_{k=0}^{T-1} (\pi_{t-1+k} - \pi_{t-1+k}^*) + \nu_t - \nu_{t-1}$$
$$- \frac{1}{T} [(\pi_t - \pi_t^*) - (\pi_{t-1} - \pi_{t-1}^*)] = \theta x_t + \varepsilon_{\bar{\nu},t},$$

where $x_t := \pi_t - \pi_t^*$ and $\varepsilon_{\bar{\nu},t} := \bar{\nu}_t - \bar{\nu}_{t-1}$. We also estimate this equation without the $\frac{1}{T}[(\pi_t - \pi_t^*) - (\pi_{t-1} - \pi_{t-1}^*)]$ term on the left-hand side.

For consistency with the information set used to price TIPS, we proxy $\mathbb{E}_t \frac{1}{T} \sum_{k=0}^{T-1} \pi_{t+k}^*$ with t measured in quarters by the final month of quarter t value of $\mathbb{E}_{t-1} \frac{1}{3T} \sum_{k=1}^{3T} \pi_{t+k-3}^*$ with t measured in months, derived from the state space model. And we use final month of quarter t values of breakeven inflation as estimates of $i_t - r_t = \mathbb{E}_t \frac{1}{T} \sum_{k=0}^{T-1} \pi_{t+k} + \nu_t$ (t quarterly). These proxies for the components of the left-hand side variable will not bias our estimates as, for example, $\mathbb{E}_{t-1} \frac{1}{3T} \sum_{k=1}^{3T} \pi_{t+k-3}^* = \mathbb{E}_t \frac{1}{3T} \sum_{k=1}^{3T} \pi_{t+k-3}^* +$ an unexpected shock (t monthly).

Figure 8 plots our estimate of $\mathbb{E}_t \frac{4}{T} \sum_{k=0}^{T-1} \pi_{t+k} + 4\nu_t$ (from observed breakeven inflation from five-year TIPS and treasuries) versus our estimate of $\mathbb{E}_t \frac{4}{T} \sum_{k=0}^{T-1} \pi_{t+k}^*$ (from the state space model). The targeted five-year inflation rate was above the breakeven one for most of the sample, perhaps reflecting an intention to make-up for the low inflation of the ZLB period, or perhaps reflecting liquidity premia on nominal bonds pushing down breakeven rates. However, since we time difference both quantities, any static wedge between the two will drop out.

See the main text for the results of estimating θ.[33] Note that in the main text, we give the fraction of variance explained of five-year breakeven inflation expectations and five-year treasury yields. These are from Federal Reserve Bank of St. Louis (2024a) and Board of Governors of the Federal Reserve System (US) (2024b).

[33] These estimates and the subsequent ones may be obtained by running the MATLAB script "Main.m" in this paper's replication materials. The code describes these regressions as the "modified" ones, and the ones without the $\frac{1}{T}[(\pi_t - \pi_t^*) - (\pi_{t-1} - \pi_{t-1}^*)]$ term on the left hand side as the unmodified ones.

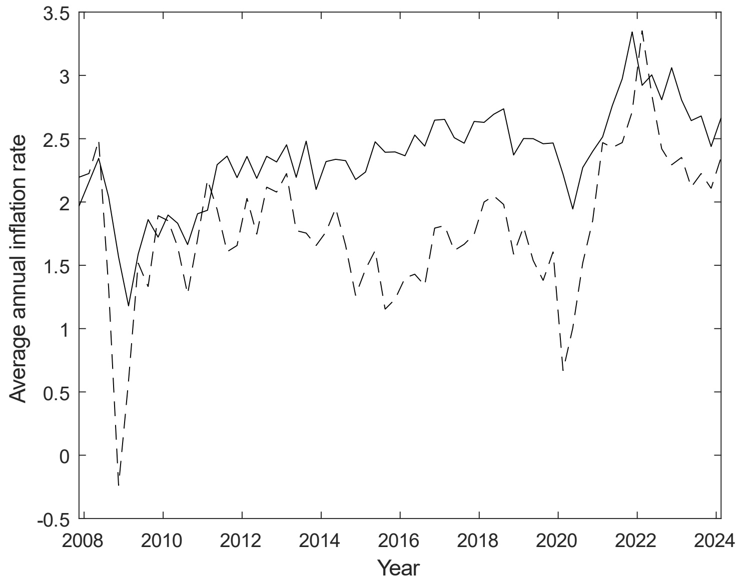

Figure 8: Estimated (smoothed) value of $\mathbb{E}_t \frac{4}{T} \sum_{k=0}^{T-1} \pi_{t+k}^{*}$ from the state space model (solid line) and $\mathbb{E}_t \frac{4}{T} \sum_{k=0}^{T-1} \pi_{t+k} + 4\nu_t$ from observed breakeven inflation (dashed line). Annualized continuously compounded rates.

Appendix K Proofs and additional results

K.1 Responding to other endogenous variables in a general model

Suppose the central bank uses the rule:

$$i_t = r_t + \phi_\pi \pi_t + \iota \phi_z^\top z_t + \phi_\nu^\top \nu_t.$$

Here, z_t is a vector of other endogenous variables, with $z_{t,1} = r_t$, $\iota > 0$ is a scalar governing the strength of response to all of them, and ν_t is an arbitrary exogenous stochastic process (potentially vector valued). As usual, we assume $\phi_\pi > 1$. We also assume without loss of generality that the elements of z_t are all zero in steady state.

Without loss of generality, we suppose that the other endogenous variables satisfy the general linear expectational difference equation:

$$0 = A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + d\pi_t + E\nu_t,$$

where the coefficient matrices are such that there is a unique matrix F with eigenvalues in the unit circle such that $F = -(AF + B)^{-1}C$.[34] This condition on F just states that there is no real indeterminacy in the model. Once inflation is

[34] The lack of terms in $\mathbb{E}_t \pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

determined, so too is z_t. Having the same shock process entering both the monetary rule and the model's other equations is without loss of generality as it is multiplied by $\phi_\nu^\top$ and E respectively.

Now define:

$$G := -A(AF + B)^{-1}.$$

Let L be the lag operator, then note that:

$$(I - GL^{-1})(AF + B)(I - FL) = AL^{-1} + B + CL.$$

Thus, by the model's real determinacy, all of G's eigenvalues must also be inside the unit circle.

In terms of the lag operator, the model to be solved is then:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t = -\iota\phi_\pi^{-1}\phi_z^\top z_t - \phi_\pi^{-1}\phi_\nu^\top \nu_t,$$
$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t = -d\pi_t - E\nu_t.$$

Note for future reference that since ϕ_π^{-1}, G and F all have all their eigenvalues in the unit circle, $(1 - \phi_\pi^{-1}L^{-1})$, $(I - GL^{-1})$ and $(I - FL)$ are all invertible.

We conjecture a series solution of the form:

$$\pi_t = \sum_{k=0}^{\infty} \iota^k \pi_t^{(k)}, \qquad z_t = \sum_{k=0}^{\infty} \iota^k z_t^{(k)}.$$

Matching terms gives that $\pi_t^{(0)}$ solves:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t^{(0)} = -\phi_\pi^{-1}\phi_\nu^\top \nu_t,$$

implying that $\pi_t^{(0)}$ is determinate with:

$$\pi_t^{(0)} = -\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_\nu^\top \nu_t.$$

Similarly, from matching terms in the law of motion for z_t, we have that:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t^{(0)} = -d\pi_t^{(0)} - E\nu_t$$

so $z_t^{(0)}$ is also determinate (by our assumption on A, B and C) with:

$$z_t^{(0)} = -(I - FL)^{-1}(AF + B)^{-1}\mathbb{E}_t(I - GL^{-1})^{-1}\big(d\pi_t^{(0)} - E\nu_t\big).$$

Note that $\pi_t^{(0)}$ can be treated as exogenous for solving for $z_t^{(0)}$, as the causation only runs one way, from $\pi_t^{(0)}$ to $z_t^{(0)}$.

Now suppose that we have established that $\pi_t^{(k)}$ and $z_t^{(k)}$ are determinate for some $k \in \mathbb{N}$, with a determined solution not a function of higher order terms. (We have already proven the base case of $k = 0$.) We seek to prove that $\pi_t^{(k+1)}$ and $z_t^{(k+1)}$ are also determinate. Matching terms again gives that:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t^{(k+1)} = -\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

so $\pi_t^{(k+1)}$ is also determinate, with:

$$\pi_t^{(k+1)} = -\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

where we used the inductive hypothesis that $z_t^{(k)}$ is already determined, and so it is effectively exogenous for the purpose of determining $\pi_t^{(k+1)}$. Then from matching terms in the law of motion for z_t:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t^{(k+1)} = -d\pi_t^{(k+1)},$$

so $z_t^{(k+1)}$ is also determinate, with:

$$z_t^{(k+1)} = -(I - FL)^{-1}(AF + B)^{-1}\mathbb{E}_t(I - GL^{-1})^{-1}d\pi_t^{(k+1)},$$

much as before. This completes our proof by induction, establishing that there is a series solution of the given form.

The only remaining thing to check is that the series does indeed converge for sufficiently small ι. This follows immediately from the product structure of the solution above, which means that the variances of $z_t^{(k)}$ and $\pi_t^{(k)}$ must be $O(h^k)$ for some $h \geq 1$. Hence for sufficiently small ι, the model is determinate. I.e., given the Taylor principle is satisfied, a sufficiently small response to other endogenous variables will not break determinacy.

K.2 Phillips curve based forecasting with ARMA(1,1) policy shocks

As before, we have the monetary rule $i_t = r_t + \phi\pi_t + \zeta_t$, which combined with the Fisher equation gives $\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t$. Suppose ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho\zeta_{t-1} + \varepsilon_{\zeta,t} + \theta\varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2)$$

with $\rho, \theta \in (-1,1)$. Then from matching coefficients, with $\phi > 1$ we have the unique solution:

$$\pi_t = -\frac{1}{\phi - \rho}\left[\zeta_t + \frac{\theta}{\phi}\varepsilon_{\zeta,t}\right].$$

Thus:

$$\pi_t - \rho\pi_{t-1} = -\frac{1}{\phi - \rho}\left(1 + \frac{\theta}{\phi}\right)\left[\varepsilon_{\zeta,t} + \frac{\phi - \rho}{\phi + \theta}\theta\varepsilon_{\zeta,t-1}\right],$$

so π_t also follows an ARMA(1,1) process. Suppose for now that $-\rho \leq \theta$, which is likely to be satisfied in reality as we expect ρ to be large and positive, while θ should be close to zero. (For example, Dotsey, Fujita & Stark (2018) find that an IMA(1,1) model fits inflation well, in which case $-\rho = -1 < \theta$ as required.) Then

$0 < \frac{\phi-\rho}{\phi+\theta} < 1$, so $\left|\frac{\phi-\rho}{\phi+\theta}\theta\right| < 1$ meaning the process for inflation is invertible. With inflation following an invertible linear process, the full-information optimal forecast of π_{t+1} is a linear combination of $\pi_t, \pi_{t-1}, \ldots$. In particular, as before x_t is not useful.

In the unlikely case in which $-\rho > \theta$, of if the forecaster's information set $\mathcal{I}_t$ is smaller than $\{\pi_t, x_t, \pi_{t-1}, x_{t-1}, \ldots\}$,[35] then x_t may contain some useful information. Combining the solution for inflation with the Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$

gives:

$$\begin{aligned} x_t &= -\frac{1}{\kappa}\left[\frac{1-\beta\rho}{\phi-\rho}\left(\zeta_t + \frac{\theta}{\phi}\varepsilon_{\zeta,t}\right) - \beta\frac{\theta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t \\ &= \frac{1}{\kappa}\left[(1-\beta\rho)\pi_t + \beta\frac{\theta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t. \end{aligned}$$

In this case, it is possible that $\mathbb{E}[\pi_{t+1}|\mathcal{I}_t] \neq \mathbb{E}[\pi_{t+1}|\mathcal{I}_{t-1}, \pi_t]$ as x_t provides an independent signal about $\varepsilon_{\zeta,t}$.

There are two important special cases. If $\omega_t = 0$, and the forecaster knows this, then:

$$\varepsilon_{\zeta,t} = \frac{\phi}{\beta\theta}[\kappa x_t - (1-\beta\rho)\pi_t],$$

so:

$$\zeta_t = -\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\kappa}{\beta}x_t,$$

which enables the forecaster to form the full-information optimal forecast:

$$\mathbb{E}_t \pi_{t+1} = -\frac{1}{\phi-\rho}\left(\rho\zeta_t + \theta\varepsilon_{\zeta,t}\right) = \frac{1}{\beta}(\pi_t - \kappa x_t).$$

(This formula also follows immediately from the Phillips curve.) Note that the output gap has what Dotsey, Fujita & Stark (2018) call the "wrong" sign, meaning Phillips curve based forecasting regressions may have surprising results. However, in the general case in which ω_t has positive variance, then output's signal about $\varepsilon_{\zeta,t}$ will be polluted by the noise from ω_t, making it much less informative. Indeed, with ϕ large, as we expect, then $\frac{\theta}{\phi}\varepsilon_{\zeta,t}$ will have low variance, making it more likely that it is drowned out by the noise from ω_t.

The second important special case is when $\varepsilon_{\zeta,t} = 0$, and again the forecaster

[35] We nonetheless assume that π_t and x_t are in $\mathcal{I}_t$.

knows this. In this case, much as in the main text:

$$\mathbb{E}_t \pi_{t+1} = \rho \pi_t - \frac{1}{\phi - \rho}\left(1 + \frac{\theta}{\phi}\right)\left[\mathbb{E}_t \varepsilon_{\zeta,t+1} + \frac{\phi - \rho}{\phi + \theta}\theta \varepsilon_{\zeta,t}\right] = \rho \pi_t,$$

so x_t is unhelpful.

The general case will inherit aspects of these two special cases, as well as the case in which π_t's stochastic process was invertible. Inflation and its lags will certainly help forecast inflation, but the output gap may also provide a little extra information, possibly with the "wrong" sign.

K.3 Determinacy under traditional price level rules

We are interested in the properties of standard Taylor-type monetary rules when augmented with a response to the price level. Suppose the model is given by equations (4) and (8), from the main text, without shocks, and with the simple monetary rule:

$$i_t = n_t + \phi \pi_t + \theta p_t + \psi x_t,$$

where p_t is the logarithm of the price level, so $\pi_t = p_t - p_{t-1}$.[36] Thus, we have the three equations:

$$\begin{aligned}\mathbb{E}_t x_{t+1} + \delta^{-1}\varsigma \mathbb{E}_t \pi_{t+1} - \delta^{-1}\varsigma \theta p_t &= \delta^{-1}(1 + \varsigma\psi) x_t + \delta^{-1}\varsigma\phi\pi_t,\\ \mathbb{E}_t \pi_{t+1} &= -\beta^{-1}\kappa x_t + \beta^{-1}\pi_t,\\ p_t &= \pi_t + p_{t-1}.\end{aligned}$$

If we subtract $\delta^{-1}\varsigma$ times the second equation from the first equation, and then add on $\delta^{-1}\varsigma\theta$ times the third, we are left with the system:

$$\mathbb{E}_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \\ p_t \end{bmatrix} = \begin{bmatrix} \delta^{-1}(1 + \varsigma\psi + \beta^{-1}\kappa\varsigma) & \delta^{-1}\varsigma(\phi + \theta - \beta^{-1}) & \delta^{-1}\varsigma\theta \\ -\beta^{-1}\kappa & \beta^{-1} & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ p_{t-1} \end{bmatrix}.$$

Determinacy requires that the matrix has one eigenvalue with modulus in $[0,1]$ (1 is included as prices need not be stationary), and two eigenvalues with modulus in $(1, \infty)$. The eigenvalues of the matrix are solutions for λ of:

$$\begin{aligned}0 = \beta\delta\lambda^3 &- [\kappa\varsigma + \delta + \beta(1 + \delta + \varsigma\psi)]\lambda^2 \\ &+ [1 + \beta + \delta + \kappa\varsigma(1 + \phi + \theta) + \varsigma\psi(1 + \beta)]\lambda - (1 + \kappa\varsigma\phi + \varsigma\psi).\end{aligned}$$

We will analyse determinacy under two alternate sets of assumptions. The first will assume that ϕ, θ and ψ are all small; the second will instead assume that $\phi -$

[36] See Footnote 3 for discussion of the validity of including the price level in this way.

1, θ and ψ are small.

Determinacy with small ϕ, θ and ψ. For the former, fix $\hat{\phi}, \hat{\theta}, \hat{\psi} \in [0, \infty)$, with $\hat{\theta} > 0$, and suppose that $\phi = \epsilon\hat{\phi}$, $\theta = \epsilon\hat{\theta}$ and $\psi = \epsilon\hat{\psi}$, where $\epsilon > 0$ is a perturbation parameter. We will work in the limit as $\epsilon \to 0$ to assess whether an arbitrarily small positive response to prices is sufficient for determinacy, as it is under a real rate rule (see Appendix H.2 in this document). Note that since we are assuming $\hat{\theta} > 0$ and $\epsilon > 0$, we always have $\theta > 0$, so there is a response to the price level. We make the following very mild assumptions in our determinacy analysis:

$$\kappa\varsigma \neq 0,$$
$$(1-\beta)(1-\delta) - \kappa\varsigma \neq 0,$$
$$(1+\beta)(1+\delta) + \kappa\varsigma > 0.$$

Given these assumptions, as $\epsilon \to 0$, we have the following solution for λ:

$$\lambda \in \{\lambda_1(\epsilon) + O(\epsilon^2), \lambda_2 + O(\epsilon), \lambda_3 + O(\epsilon)\},$$

where for all ϵ:

$$\lambda_1(\epsilon) = 1 - \frac{\kappa\varsigma}{(1-\beta)(1-\delta) - \kappa\varsigma}\hat{\theta}\epsilon,$$

and where λ_2 and λ_3 solve $0 = f(\lambda_2) = f(\lambda_3)$, where the function f is defined by:

$$f(\lambda) = \beta\delta\lambda^2 - (\kappa\varsigma + \beta + \delta)\lambda + 1,$$

for all λ. Without loss of generality, we assume $|\lambda_3| \geq |\lambda_2|$.

We now distinguish two cases.

Case 1: Firstly, suppose that:

$$\frac{(1-\beta)(1-\delta) - \kappa\varsigma}{\kappa\varsigma} < 0.$$

Then for all $\epsilon > 0$, $\lambda_1(\epsilon) > 1$, so we must have that $|\lambda_3| > 1$ and $|\lambda_2| \leq 1$. Note also that λ_2 and λ_3 must be real, else they would be complex conjugates and hence have equal modulus, contradicting $|\lambda_3| > 1 \geq |\lambda_2|$.

Now, note that $f(0) = 1 > 0$, and $f(-1) = (1+\beta)(1+\delta) + \kappa\varsigma > 0$ (by our assumption), so there cannot be a single root in the interval $(-1,0)$. Since there cannot be two, in fact there must be zero. Thus, it must be the case that $f(1) \leq 0$, else there would be zero or two roots in $(0,1]$. So, $f(1) = (1-\beta)(1-\delta) - \kappa\varsigma \leq 0$. But $(1-\beta)(1-\delta) - \kappa\varsigma \neq 0$, so in fact $(1-\beta)(1-\delta) - \kappa\varsigma < 0$. Hence, $\kappa\varsigma > 0$, as $\frac{(1-\beta)(1-\delta)-\kappa\varsigma}{\kappa\varsigma} < 0$ in the currently considered case. Given $\kappa\varsigma > 0$, we are then guaranteed that $\lambda_2 \in (0,1)$ and $\lambda_3 \in (1, \infty)$, as required for determinacy.

Case 2: Now suppose instead that:

$$\frac{(1-\beta)(1-\delta)-\kappa\varsigma}{\kappa\varsigma} > 0.$$

(Note that we do not have to consider the equality case as $(1-\beta)(1-\delta)-\kappa\varsigma \neq 0$ by assumption.) Then, $\lambda_1 < 1$ so we must have that $1 < |\lambda_2| \leq |\lambda_3|$.

Much as before, $f(0) = 1 > 0$, and $f(-1) = (1+\beta)(1+\delta)+\kappa\varsigma > 0$ (by assumption), so we must have that $f(1) = (1-\beta)(1-\delta)-\kappa\varsigma > 0$, else there would be a root in the unit circle. This implies that $\kappa\varsigma > 0$ as $\frac{(1-\beta)(1-\delta)-\kappa\varsigma}{\kappa\varsigma} > 0$ in the currently considered case.

Therefore, in either case, $\kappa\varsigma > 0$ is necessary for determinacy.

Determinacy with small $\phi - 1$, θ and ψ. We now consider the case in which rather than ϕ being small and non-negative, instead $\phi - 1$ is small and non-negative, so there is at least a unit response to inflation. Some researchers include at least a unit response to inflation in price level rules (for example, Bernanke, Kiley & Roberts (2019)).

Then, as before, we fix $\hat{\phi}, \hat{\theta}, \hat{\psi} \in [0, \infty)$, with $\hat{\theta} > 0$, and suppose that $\phi = 1 + \epsilon\hat{\phi}$, $\theta = \epsilon\hat{\theta}$ and $\psi = \epsilon\hat{\psi}$, where $\epsilon > 0$ is a perturbation parameter. We will again work in the limit as $\epsilon \to 0$ to assess whether an arbitrarily small positive response to prices is sufficient for determinacy, as it is under a real rate rule (see Appendix H.2 in this document). Note that since we are assuming $\hat{\theta} > 0$ and $\epsilon > 0$, we always have $\theta > 0$, so there is a response to the price level. We make the following quite mild assumptions in our determinacy analysis:

$$\begin{gathered}
\kappa\varsigma \neq 0, \\
(1-\beta)(1-\delta) \neq 0, \\
1+\kappa\varsigma > 0, \\
(1+\beta)(1+\delta)+2\kappa\varsigma > 0.
\end{gathered}$$

(Note that $(1-\beta)(1-\delta) \neq 0$ does rule out the classical NK model with $\delta = 1$, but only an arbitrarily small departure from this benchmark is needed for our results to go through.)

Given these assumptions, as $\epsilon \to 0$, we have the following solution for λ:

$$\lambda \in \{\lambda_1(\epsilon) + \mathrm{O}(\epsilon^2), \lambda_2 + \mathrm{O}(\epsilon), \lambda_3 + \mathrm{O}(\epsilon)\},$$

where for all ϵ:

$$\lambda_1(\epsilon) = 1 - \frac{\kappa\varsigma}{(1-\beta)(1-\delta)}\hat{\theta}\epsilon,$$

and where λ_2 and λ_3 solve $0 = f(\lambda_2) = f(\lambda_3)$, where the function f is defined by:

$$f(\lambda) = \beta\delta\lambda^2 - (\kappa\varsigma + \beta + \delta)\lambda + 1 + \kappa\varsigma,$$

for all λ. Without loss of generality, we assume $|\lambda_3| \geq |\lambda_2|$.

We now distinguish two cases.

Case 1: Firstly, suppose that:

$$\frac{(1-\beta)(1-\delta)}{\kappa\varsigma} < 0.$$

Then for all $\epsilon > 0, \lambda_1(\epsilon) > 1$, so we must have that $|\lambda_3| > 1$ and $|\lambda_2| \leq 1$. Note also that λ_2 and λ_3 must be real, else they would be complex conjugates and hence have equal modulus, contradicting $|\lambda_3| > 1 \geq |\lambda_2|$.

Now, note that $f(0) = 1 + \kappa\varsigma > 0$, and $f(-1) = (1+\beta)(1+\delta) + 2\kappa\varsigma > 0$ (by our assumptions), so there cannot be a single root in the interval $(-1,0)$. Since there cannot be two, in fact there must be zero. Thus, it must be the case that $f(1) \leq 0$, else there would be zero or two roots in $(0,1]$. So, $f(1) = (1-\beta)(1-\delta) \leq 0$. But $(1-\beta)(1-\delta) \neq 0$, so in fact $(1-\beta)(1-\delta) < 0$. Hence, $\kappa\varsigma > 0$, as $\frac{(1-\beta)(1-\delta)}{\kappa\varsigma} < 0$ in the currently considered case. Given $\kappa\varsigma > 0$, we are then guaranteed that $\lambda_2 \in (0,1)$ and $\lambda_3 \in (1,\infty)$, as required for determinacy.

Case 2: Now suppose instead that:

$$\frac{(1-\beta)(1-\delta)}{\kappa\varsigma} > 0.$$

(Note that we do not have to consider the equality case as $(1-\beta)(1-\delta) \neq 0$ by assumption.) Then, $\lambda_1 < 1$ so we must have that $1 < |\lambda_2| \leq |\lambda_3|$.

Much as before, $f(0) = 1 + \kappa\varsigma > 0$, and $f(-1) = (1+\beta)(1+\delta) + 2\kappa\varsigma > 0$ (by assumption), so we must have that $f(1) = (1-\beta)(1-\delta) > 0$, else there would be a root in the unit circle. This implies that $\kappa\varsigma > 0$ as $\frac{(1-\beta)(1-\delta)-\kappa\varsigma}{\kappa\varsigma} > 0$ in the currently considered case.

Therefore, again, in either case, $\kappa\varsigma > 0$ is necessary for determinacy.

K.4 Robustness to non-unit responses to real interest rates

Suppose that the central bank is unable to respond with a precise unit coefficient to real interest rates, so instead follows the monetary rule:

$$i_t = (1+\gamma)r_t + \phi\pi_t + \zeta_t,$$

where $\gamma \in \mathbb{R}$ is some small value giving the departure from unit responses.

For simplicity, suppose the rest of the model takes the same form as in Subsection 3.2, with:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \varsigma (r_t - n_t),$$
$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$
$$i_t = r_t + \mathbb{E}_t \pi_{t+1}.$$

We suppose $\phi > 1$, but do not make any assumptions on the signs of $\delta, \beta, \kappa, \varsigma, \gamma$, beyond assuming that $\varsigma \neq 0$ (so monetary policy has some effect on the output gap) and $\kappa \neq 0$ (so monetary policy has some effect on inflation, via the output gap).

Combining the monetary rule with the Fisher equation gives:

$$\mathbb{E}_t \pi_{t+1} = \gamma r_t + \phi \pi_t + \zeta_t,$$

so $r_t = \frac{1}{\gamma} (\mathbb{E}_t \pi_{t+1} - \phi \pi_t - \zeta_t)$, meaning:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \frac{\varsigma}{\gamma} (\mathbb{E}_t \pi_{t+1} - \phi \pi_t) + \varsigma n_t + \frac{\varsigma}{\gamma} \zeta_t.$$

Then, since:

$$\mathbb{E}_t \pi_{t+1} = \frac{1}{\beta} \pi_t - \frac{\kappa}{\beta} x_t - \frac{\kappa}{\beta} \omega_t,$$

we have that:

$$\mathbb{E}_t x_{t+1} = \left(\frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} \right) x_t - \frac{\varsigma}{\delta \gamma} \left(\phi - \frac{1}{\beta} \right) \pi_t - \frac{\varsigma}{\delta \gamma} \left(\gamma n_t + \zeta_t + \frac{\kappa}{\beta} \omega_t \right).$$

Woodford (2003) (Addendum to Chapter 4, Proposition C.1) proves that this model is determinate if and only if both eigenvalues of the matrix:

$$M := \begin{bmatrix} \frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} & -\frac{\varsigma}{\delta \gamma} \left(\phi - \frac{1}{\beta} \right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

are outside of the unit circle, which in turn is proven to hold if and only if EITHER: Case I: $1 < \det M$, $0 < 1 + \det M - \operatorname{tr} M$, and $0 < 1 + \det M + \operatorname{tr} M$, OR Case II: $0 > 1 + \det M - \operatorname{tr} M$, and $0 > 1 + \det M + \operatorname{tr} M$. Note:

$$\det M = \frac{1}{\beta \delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} \phi,$$
$$\operatorname{tr} M = \frac{1}{\delta} - \frac{\varsigma \kappa}{\gamma \beta \delta} + \frac{1}{\beta}.$$

Thus, Case I requires:

$$1 < \det M = \frac{1}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\phi,$$
$$0 < 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$
$$\text{and } 0 < 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

And Case II requires:

$$0 > 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$
$$\text{and } 0 > 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

To see when these conditions are satisfied, first suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} < 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = -\frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then if γ is sufficiently small in magnitude, it is immediately clear that all three conditions of Case I are satisfied, since $\phi > 0$, $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{\max\{0, -(\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, -(\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.$$

Alternatively, suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} > 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = \frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then, similarly, if γ is sufficiently small in magnitude, both conditions of Case II are satisfied, since $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi - 1}{\max\{0, (\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1 + \phi}{\max\{0, (\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{array} \right\}.$$

Thus, it is always sufficient for determinacy that:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{array}{c} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{|(1-\beta)(1-\delta)|}, \\ \dfrac{1 + \phi}{|(1+\beta)(1+\delta)|} \end{array} \right\}.$$

Since the right-hand side is strictly positive, there is a positive measure of γ for which we have determinacy.

K.5 Real-time learning of Phillips curve coefficients

We start by assuming that the central bank knows the Phillips curve coefficients. A close examination of this case will lead to a natural learning scheme for when the central bank does not know these coefficients.

As in the main text, suppose the central bank is using the rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \Big[x_t - \kappa^{-1} \big[\pi_t - \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} - \tilde{\beta} \varrho_\pi \pi_{t-1} \big] \Big] + \zeta_t,$$

and that the model also contains the Phillips curve:

$$\pi_t = \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} + \tilde{\beta} \varrho_\pi \pi_{t-1} + \kappa x_t + \kappa \omega_t,$$

and the Fisher equation, $i_t = r_t + \mathbb{E}_t \pi_{t+1}$.

We suppose that ζ_t follows the ARMA$(1{,}1)$ process:

$$\zeta_t = \rho \zeta_{t-1} + \varepsilon_{\zeta,t} + \theta \varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N\big(0, \sigma_\zeta^2\big),$$

with $\rho, \theta \in (-1{,}1)$, and for simplicity, we suppose that $\omega_t = \varepsilon_{\omega,t}$, where $\varepsilon_{\omega,t} \sim N(0, \sigma_\omega^2)$.

From combining all the above equations, we have that if $\phi_\pi > 1$, there is a unique solution with:

$$\pi_t = -\frac{1}{\phi_\pi - \rho} \left[\zeta_t + \frac{\theta}{\phi_\pi} \varepsilon_{\zeta,t} \right] + \frac{\phi_x}{\phi_\pi} \varepsilon_{\omega,t}.$$

Thus, if we define:

$$m_0 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho)} \left[\tilde{\beta}(1 - \varrho_\pi)(\rho + \theta) - \left(1 + \frac{\theta}{\phi_\pi} \right) \right],$$

$$m_1 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho)} \left[\big[\tilde{\beta}(1 - \varrho_\pi)\rho - 1 \big](\rho + \theta) + \tilde{\beta} \varrho_\pi \left(1 + \frac{\theta}{\phi_\pi} \right) \right],$$

$$m_2 := \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho)} \big[\big[\tilde{\beta}(1 - \varrho_\pi)\rho - 1 \big] \rho + \tilde{\beta} \varrho_\pi \big](\rho + \theta),$$

then by the Phillips curve $m_0 = \mathbb{E} x_t \varepsilon_{\zeta,t}$, $m_1 = \mathbb{E} x_t \varepsilon_{\zeta,t-1}$ and $m_2 = \mathbb{E} x_t \varepsilon_{\zeta,t-2}$. Also note that:

$$\kappa = \frac{\sigma_\zeta^2}{\phi_\pi - \rho} \frac{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right)^2}{\rho\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right) m_0 - \left((\rho + \theta) m_1 - \left(1 + \frac{\theta}{\phi_\pi}\right) m_2 \right)},$$

$$\beta = \frac{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right)\left(m_0 - (\rho m_1 - m_2) \right) - \frac{\phi_\pi + \theta}{(\rho + \theta)\phi_\pi} \left((\rho + \theta) m_1 - \left(1 + \frac{\theta}{\phi_\pi}\right) m_2 \right)}{\rho\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right) m_0 - \left((\rho + \theta) m_1 - \left(1 + \frac{\theta}{\phi_\pi}\right) m_2 \right)},$$

$$\varrho_\pi = -\frac{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right)(\rho m_1 - m_2)}{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho \right)\left(m_0 - (\rho m_1 - m_2) \right) - \frac{\phi_\pi + \theta}{(\rho + \theta)\phi_\pi} \left((\rho + \theta) m_1 - \left(1 + \frac{\theta}{\phi_\pi}\right) m_2 \right)}.$$

In other words, once the central bank knows m_0, m_1 and m_2 they can infer the parameters of the Phillips curve from the known properties of their monetary rule and monetary shock. This is essentially an instrumental variables regression. We are using $\varepsilon_{\zeta,t}$, $\varepsilon_{\zeta,t-1}$ and $\varepsilon_{\zeta,t-2}$ as instruments for $\mathbb{E}_t\pi_{t+1}$, π_t and π_{t-1} in a regression of the output gap on those variables. This works as long as $\theta \neq 0$, else $\mathbb{E}_t\pi_{t+1}$ and π_t are colinear.

If the central bank does not know the true values of κ, $\tilde{\beta}$ and ϱ_π, we suppose they dynamically update estimates of m_0, m_1 and m_2 using the following decreasing gain learning rules (for $t > 0$):

$$\begin{aligned} m_{0,t} &= m_{0,t-1} + t^{-1}\big(x_t\varepsilon_{\zeta,t} - m_{0,t-1}\big), \\ m_{1,t} &= m_{1,t-1} + t^{-1}\big(x_t\varepsilon_{\zeta,t-1} - m_{1,t-1}\big), \\ m_{2,t} &= m_{2,t-1} + t^{-1}\big(x_t\varepsilon_{\zeta,t-2} - m_{2,t-1}\big), \end{aligned}$$

where $\iota \in (0,1]$ is a gain parameter. Then they can use the monetary rule:

$$i_t = r_t + \phi_\pi\pi_t + \phi_x[x_t + q_{1,t-1}\mathbb{E}_t\pi_{t+1} + q_{0,t-1}\pi_t + q_{-1,t-1}\pi_{t-1}] + \zeta_t,$$

where:

$$\begin{aligned} q_{1,t} &:= \frac{\phi_\pi-\rho}{\sigma_\zeta^2}\frac{\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)m_{0,t}-\frac{\phi_\pi+\theta}{(\rho+\theta)\phi_\pi}\Big((\rho+\theta)m_{1,t}-\big(1+\frac{\theta}{\phi_\pi}\big)m_{2,t}\Big)}{\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)^2}, \\ q_{0,t} &:= -\frac{\phi_\pi-\rho}{\sigma_\zeta^2}\frac{\rho\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)m_{0,t}-\Big((\rho+\theta)m_{1,t}-\big(1+\frac{\theta}{\phi_\pi}\big)m_{2,t}\Big)}{\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)^2}, \\ q_{-1,t} &:= -\frac{\phi_\pi-\rho}{\sigma_\zeta^2}\frac{\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)(\rho m_{1,t}-m_{2,t})}{\big(\rho+\theta-\big(1+\frac{\theta}{\phi_\pi}\big)\rho\big)^2}. \end{aligned}$$

This is reasonable, as if $m_{0,t-1} \approx m_0$, $m_{1,t-1} \approx m_1$ and $m_{2,t-1} \approx m_2$ then $q_{1,t-1} \approx \kappa^{-1}\tilde{\beta}(1-\varrho_\pi)$, $q_{0,t-1} \approx -\kappa^{-1}$ and $q_{-1,t-1} \approx \kappa^{-1}\tilde{\beta}\varrho_\pi$, so this monetary rule is approximately the same as the full information one previously considered. Using lagged estimates ($q_{1,t-1}$ not $q_{1,t}$ etc.) in the monetary rule reflects central bank information (processing) delays and simplifies the model's solution. It is also a common assumption in the reduced form learning literature (Evans & Honkapohja 2001).

With the new monetary rule, the model is no-longer linear. As a result, the exact solution is analytically intractable. However, we are only really interested in asymptotic dynamics. If $m_{0,t} \to m_0$, $m_{1,t} \to m_1$ and $m_{2,t} \to m_2$ as $t \to \infty$ then we know the asymptotic solution will be the stable full information one we found previously. We will analyse the system's behaviour with help from the stochastic approximation tools frequently used in the reduced form learning literature (Evans & Honkapohja 2001). These tools only require a zeroth order approximation in t^{-1} to the dynamics of x_t and π_t.[37] Intuitively, this is because x_t (hence π_t) enters the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ multiplied by t^{-1}, so a zeroth order approximation to the dynamics of x_t and π_t in t^{-1} delivers a first order approximation to the dynamics of $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ in t^{-1}.

We conjecture a time-varying coefficients solution with:

$$\pi_t = A_{t-1}\zeta_t + B_{t-1}\varepsilon_{\zeta,t} + C_{t-1}\varepsilon_{\omega,t} + D_{t-1}\pi_{t-1} + O(t^{-1}),$$

where we conjecture $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$. Substituting this into the monetary rule, Fisher equation and Phillips curve implies:

$$\begin{aligned}
&\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t-1}\right]A_t\left(\rho\zeta_t + \theta\varepsilon_{\zeta,t}\right)\\
&\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t-1}\right.\\
&\qquad \left. - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t-1}\right]D_t\right]\left[A_{t-1}\zeta_t + B_{t-1}\varepsilon_{\zeta,t}\right.\\
&\qquad \left. + C_{t-1}\varepsilon_{\omega,t} + D_{t-1}\pi_{t-1}\right] + \phi_x\left[q_{-1,t-1} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right]\pi_{t-1} - \phi_x\varepsilon_{\omega,t} + \zeta_t\\
&\qquad + O(t^{-1}).
\end{aligned}$$

Matching terms and using $A_t = A_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$ then gives that:

$$\begin{aligned}
&\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\right]A_t\rho\\
&\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]A_t\\
&\qquad + 1 + O(t^{-1}),\\
&\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\right]A_t\theta\\
&\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]B_t\\
&\qquad + O(t^{-1}),\\
&0 = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t-1}\right]D_t\right]C_t - \phi_x\\
&\qquad + O(t^{-1}),\\
&0 = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]D_t\\
&\qquad + \phi_x\left[q_{-1,t} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right] + O(t^{-1}).
\end{aligned}$$

The final equation has two roots, but we know we need to pick the one that gives

[37] Given certain regularity conditions on the higher order terms. These conditions will be satisfied here, at least providing we restrict $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ to a small enough open set around m_0, m_1 and m_2, using a so called projection facility.

$D_t \to 0$ as $\phi_x \to 0$. Now if $q_{0,t}$ is sufficiently close to q_0, then $\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t} > 0$, so:

$$\begin{aligned} D_t = {} & \frac{(\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t}) - \sqrt{\begin{matrix} (\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t})^2 \cdots \\ +4\phi_x \big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big]\big[q_{-1,t} - \kappa^{-1}\tilde{\beta}\varrho_\pi\big] \end{matrix}}}{2\big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big]} \\ & + O(t^{-1}), \end{aligned}$$

and:

$$\begin{aligned} A_t &= \Big[\big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big](D_t + \rho) - (\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t})\Big]^{-1} \\ &\qquad + O(t^{-1}), \\ B_t &= \frac{\theta\big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big] A_t}{\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t} - \big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big] D_t} + O(t^{-1}), \\ C_t &= \frac{\phi_x}{\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t} - \big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\big] D_t} + O(t^{-1}). \end{aligned}$$

Since $q_{1,t} = q_{1,t-1} + O(t^{-1})$, $q_{0,t} = q_{0,t-1} + O(t^{-1})$ and $q_{-1,t} = q_{-1,t-1} + O(t^{-1})$, as required we have that $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$.

Using this result again, we then have that:

$$\begin{aligned} x_t = \kappa^{-1} \Big[& \big[1 - \tilde{\beta}(1-\varrho_\pi)(D_{t-1} + \rho)\big] A_{t-1} \zeta_t \\ & + \big[B_{t-1} - \tilde{\beta}(1-\varrho_\pi)(A_{t-1}\theta + B_{t-1} D_{t-1})\big] \varepsilon_{\zeta,t} \\ & + \big[\big[1 - \tilde{\beta}(1-\varrho_\pi) D_{t-1}\big] C_{t-1} - \kappa\big] \varepsilon_{\omega,t} \\ & + \big[\big[1 - \tilde{\beta}(1-\varrho_\pi) D_{t-1}\big] D_{t-1} - \tilde{\beta}\varrho_\pi\big] \pi_{t-1} \Big] + O(t^{-1}). \end{aligned}$$

Plugging this into the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ gives a purely backward looking non-linear system in the endogenous states $m_{0,t}$, $m_{1,t}$, $m_{2,t}$ and π_t. This system is of the correct form to be analysed by the stochastic approximation results given in Evans & Honkapohja (2001).

To apply these results, first suppose that for all t, $m_{0,t} = \widehat{m}_0$, $m_{1,t} = \widehat{m}_1$ and $m_{2,t} = \widehat{m}_2$, for some values $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$. Then $q_{1,t} = \hat{q}_1$, $q_{0,t} = \hat{q}_0$ and $q_{-1,t} = \hat{q}_{-1}$ for all t, where:

$$\begin{aligned} \hat{q}_1 &:= \frac{\phi_\pi - \rho}{\sigma_\zeta^2} \frac{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho\right)\widehat{m}_0 - \frac{\phi_\pi + \theta}{(\rho+\theta)\phi_\pi}\left((\rho+\theta)\widehat{m}_1 - \left(1 + \frac{\theta}{\phi_\pi}\right)\widehat{m}_2\right)}{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho\right)^2}, \\ \hat{q}_0 &:= -\frac{\phi_\pi - \rho}{\sigma_\zeta^2} \frac{\rho\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho\right)\widehat{m}_0 - \left((\rho+\theta)\widehat{m}_1 - \left(1 + \frac{\theta}{\phi_\pi}\right)\widehat{m}_2\right)}{\left(\rho + \theta - \left(1 + \frac{\theta}{\phi_\pi}\right)\rho\right)^2}, \end{aligned}$$

$$\hat{q}_{-1} := -\frac{\phi_\pi - \rho}{\sigma_\zeta^2} \frac{\left(\rho+\theta-\left(1+\frac{\theta}{\phi_\pi}\right)\rho\right)(\rho\widehat{m}_1 - \widehat{m}_2)}{\left(\rho+\theta-\left(1+\frac{\theta}{\phi_\pi}\right)\rho\right)^2}.$$

Thus, for all t, $A_t = \hat{A}$, $B_t = \hat{B}$, $C_t = \hat{C}$ and $D_t = \widehat{D}$, where:

$$\widehat{D} = \frac{(\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0) - \sqrt{\begin{array}{c}(\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0)^2 \cdots \\ +4\phi_x\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]\left[\hat{q}_{-1} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right]\end{array}}}{2\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]},$$

and:

$$\hat{A} = \left[\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]\left(\widehat{D} + \rho\right) - (\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0)\right]^{-1},$$

$$\hat{B} = \frac{\theta\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]\hat{A}}{\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0 - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]\widehat{D}},$$

$$\hat{C} = \frac{\phi_x}{\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0 - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1 - \varrho_\pi) - \phi_x\hat{q}_1\right]\widehat{D}}.$$

So:

$$\pi_t = \hat{A}\zeta_t + \hat{B}\varepsilon_{\zeta,t} + \hat{C}\varepsilon_{\omega,t} + \widehat{D}\pi_{t-1},$$

and:

$$\begin{aligned}
x_t = \kappa^{-1}\Bigg[&\left[1 - \tilde{\beta}(1 - \varrho_\pi)\left(\widehat{D} + \rho\right)\right]\hat{A}\zeta_t + \left[\hat{B} - \tilde{\beta}(1 - \varrho_\pi)\left(\hat{A}\theta + \hat{B}\widehat{D}\right)\right]\varepsilon_{\zeta,t} \\
&+ \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\hat{C} - \kappa\right]\varepsilon_{\omega,t} \\
&+ \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\widehat{D} - \tilde{\beta}\varrho_\pi\right]\pi_{t-1}\Bigg] \\
= \kappa^{-1}\Bigg[&\left[1 - \tilde{\beta}(1 - \varrho_\pi)\left(\widehat{D} + \rho\right)\right]\hat{A}\left[\rho\left[\rho\zeta_{t-2} + \varepsilon_{\zeta,t-1} + \theta\varepsilon_{\zeta,t-2}\right] + \varepsilon_{\zeta,t} + \theta\varepsilon_{\zeta,t-1}\right] \\
&+ \left[\hat{B} - \tilde{\beta}(1 - \varrho_\pi)\left(\hat{A}\theta + \hat{B}\widehat{D}\right)\right]\varepsilon_{\zeta,t} + \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\hat{C} - \kappa\right]\varepsilon_{\omega,t} \\
&+ \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\widehat{D} - \tilde{\beta}\varrho_\pi\right]\Big[\hat{A}\left[\rho\zeta_{t-2} + \varepsilon_{\zeta,t-1} + \theta\varepsilon_{\zeta,t-2}\right] \\
&+ \hat{B}\varepsilon_{\zeta,t-1} + \hat{C}\varepsilon_{\omega,t-1} + \widehat{D}\left[\hat{A}\zeta_{t-2} + \hat{B}\varepsilon_{\zeta,t-2} + \hat{C}\varepsilon_{\omega,t-2} + \widehat{D}\pi_{t-3}\right]\Big]\Bigg].
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathbb{E}x_t\varepsilon_{\zeta,t} &= \sigma_\zeta^2\kappa^{-1}\left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\left(\widehat{D} + \rho + \theta\right)\right]\hat{A} + \left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\hat{B}\right], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-1} &= \sigma_\zeta^2\kappa^{-1}\Bigg[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\left(\widehat{D} + \rho\right)\right]\hat{A}(\rho + \theta) \\
&\quad + \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\widehat{D} - \tilde{\beta}\varrho_\pi\right]\left(\hat{A} + \hat{B}\right)\Bigg], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-2} &= \sigma_\zeta^2\kappa^{-1}\Bigg[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\left(\widehat{D} + \rho\right)\right]\hat{A}\rho(\rho + \theta) \\
&\quad + \left[\left[1 - \tilde{\beta}(1 - \varrho_\pi)\widehat{D}\right]\widehat{D} - \tilde{\beta}\varrho_\pi\right]\left[\hat{A}(\rho + \theta) + \widehat{D}\left(\hat{A} + \hat{B}\right)\right]\Bigg].
\end{aligned}$$

Now denote by $\mathcal{T}$ the map taking the vector:

$$\widehat{m} := \begin{bmatrix}\widehat{m}_0 \\ \widehat{m}_1 \\ \widehat{m}_2\end{bmatrix}$$

to the vector:

$$\mathcal{T}(\widehat{m}) := \begin{bmatrix} \mathbb{E}x_t\varepsilon_{\zeta,t} \\ \mathbb{E}x_t\varepsilon_{\zeta,t-1} \\ \mathbb{E}x_t\varepsilon_{\zeta,t-2} \end{bmatrix}.$$

Stochastic approximation theory relates the stability of our nonlinear difference equation to the stability of the ODE:

$$\frac{d\widehat{m}(\tau)}{d\tau} = \mathcal{T}\left(\widehat{m}(\tau)\right) - \widehat{m}(\tau).$$

The $\mathcal{T}$ map here plays the role usually played by the mapping from the perceived law of motion to the actual law of motion in the reduced form learning literature (Evans & Honkapohja 2001).

We conjecture that:

$$m := \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}$$

is a locally asymptotically stable point of this ODE. To check this, note that tedious algebra gives that:

$$\left.\frac{\partial\mathcal{T}(\widehat{m})}{\partial\widehat{m}}\right|_{\widehat{m}=m} = \frac{\phi_x}{\kappa\phi_\pi}\begin{bmatrix} 1 & \phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi) & \dfrac{\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)}{\phi_\pi - \rho} \\ -\tilde{\beta}\varrho_\pi & 1 - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\left[\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)\right] - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi}{\phi_\pi - \rho} \\ 0 & -\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\left[1 - \tilde{\beta}(1-\varrho_\pi)\rho\right] - \tilde{\beta}\varrho_\pi}{\phi_\pi - \rho} \end{bmatrix}.$$

For simplicity, we assume $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \geq 0$, $\varrho_\pi \in [0,1)$, $\rho \in [0,1)$ and $\phi_\pi \geq \left[\tilde{\beta}(1-\varrho_\pi)\right]^{-1}$. Under these assumptions, the off-diagonal elements of this matrix are all non-positive. Other cases may also go through, but for the sake of brevity we concentrate on this most relevant case. Given these assumptions, applying the Gershgorin circle theorem to the columns of this matrix gives the following upper bound on the real part of the eigenvalues of $\left.\frac{\partial\mathcal{T}(\widehat{m})}{\partial\widehat{m}}\right|_{\widehat{m}=m}$:

$$\frac{\phi_x}{\kappa\phi_\pi}\max\left\{\begin{matrix} 1 + \tilde{\beta}\varrho_\pi, \phi_\pi^{-1}\left[\tilde{\beta}(\phi_\pi - \varrho_\pi) + \phi_\pi - 1\right], \\ \dfrac{(1-\phi_\pi^{-1})(\phi_\pi - \tilde{\beta}\varrho_\pi) + \tilde{\beta}(1-\varrho_\pi)[1 + \phi_\pi(1-\rho)] - \phi_\pi^{-1}}{\phi_\pi - \rho} \end{matrix}\right\}.$$

The first and second arguments in curly brackets here are both less than $1 + \tilde{\beta}$. Taking the derivative of the third argument in curly brackets with respect to ρ produces an expression whose sign is not a function of ρ. Thus, the third argument in curly brackets is maximized at either $\rho = 0$ or $\rho = 1$. In the former case, the argument is less or equal to $1 + \tilde{\beta}$ providing $\tilde{\beta} \le 1$. In the latter case, the argument is less or equal to $1 + \tilde{\beta}$ providing that $2(1 - \varrho_\pi) \le \phi_\pi$. Therefore, if $\phi_x \ge 0$, $\phi_\pi \ge 0$, $\kappa \ge 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho \in [0,1)$ and:

$$\phi_\pi > \max\left\{\frac{1}{\tilde{\beta}(1 - \varrho_\pi)}, 2(1 - \varrho_\pi), \frac{\phi_x(1 + \tilde{\beta})}{\kappa}\right\},$$

then all of the eigenvalues of $\left.\frac{\partial \mathcal{T}(\widehat{m})}{\partial \widehat{m}}\right|_{\widehat{m}=m}$ are less than one. Consequently, in this case the ODE is locally asymptotically stable, so the stochastic approximation results of Evans & Honkapohja (2001) apply. In particular, if we suppose that $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$ are constrained to remain within a sufficiently small ball around m_0, m_1 and m_2, then the central bank's estimates of the Phillips curve parameters will converge to their true values, and the model's dynamics will converge to the determinate ones under rational expectations.

K.6 Real rate rules with exogenous targets

We want to prove that even with an exogenous π_t^*, rules in the form of (7) can still mimic the outcomes of any other monetary policy regime.

Suppose that the central bank were to set interest rates in a different (though time invariant) way, for example by using another rule, or by adopting optimal policy under either commitment or discretion, given some objective. For simplicity, suppose further that the economy's equilibrium conditions are linear, e.g., because we are working under a first order approximation. Let $(\varepsilon_{1,t}, \dots, \varepsilon_{N,t})_{t\in\mathbb{Z}}$ be the set of structural shocks in the economy,[38] all of which are assumed mean zero and independent both of each other, and over time. Finally, assume that the central bank's behaviour produces stationary inflation, $\tilde{\pi}_t$, with the $\tilde{\square}$ denoting that this is inflation under the alternative monetary regime. Then, by linearity and stationarity, there must exist a constant $\tilde{\pi}^*$ and coefficients $(\theta_{1,k}, \dots, \theta_{N,k})_{k\in\mathbb{N}}$ such that:

$$\tilde{\pi}_t = \tilde{\pi}^* + \sum_{k=0}^{\infty}\sum_{n=1}^{N} \theta_{n,k}\varepsilon_{n,t-k},$$

with $\sum_{k=0}^{\infty} \theta_{n,k}^2 < \infty$ for $n = 1, \dots, N$. So, if the central bank sets:

[38] This may include sunspot shocks if they are added following Farmer, Khramov & Nicolò (2015).

$$\pi_t^* = \tilde{\pi}^* + \sum_{k=0}^{\infty}\sum_{n=1}^{N} \theta_{n,k}\varepsilon_{n,t-k},$$

(exogenous!) and uses the rule (7), then for all t and in all states of the world, $\pi_t = \pi_t^* = \tilde{\pi}_t$. Moreover, this implies in turn that all the endogenous variables in the two economies must be identical in all periods and in all states of the world.

To see this final claim, let z_t and $\tilde{z}_t$ be vectors stacking the endogenous variables other than inflation in the economy with our rule and the economy with the alternative rule, respectively, with $z_{t,1} = r_t$ and $\tilde{z}_{t,1} = \tilde{r}_t$. We assume without loss of generality that the elements of z_t and $\tilde{z}_t$ are all zero in steady state.

By linearity, without loss of generality, the equations other than the monetary rule or monetary policy first order condition must have the form:[39]

$$0 = A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + d\pi_t + \sum_{n=1}^{N} f_n\varepsilon_{n,t}\,, \tag{18}$$

in the economy with our rule, and they must have the form:

$$0 = A\mathbb{E}_t\tilde{z}_{t+1} + B\tilde{z}_t + C\tilde{z}_{t-1} + d\tilde{\pi}_t + \sum_{n=1}^{N} f_n\varepsilon_{n,t},$$

in the economy with the alternative rule. (Here, A, B and C are square matrices, while d and $f_1,\dots,f_N$ are vectors.) Since $\pi_t = \tilde{\pi}_t$ for all t, $z_t = \tilde{z}_t$ must solve equation (18) for all t. It will be the unique solution providing the model has no source of indeterminacy other than perhaps monetary policy. For example, in a three equation NK model, given that $\pi_t \equiv \tilde{\pi}_t$, the Phillips curve implies that the output gap must agree in the two economies, thus the Euler equation then implies that the interest rate must also agree.

To see the uniqueness more formally, suppose that there is a unique matrix F with eigenvalues in the unit circle such that $F = -(AF+B)^{-1}C$. This condition on F just states that there is no real indeterminacy in the model.

Now define:

$$G := -A(AF+B)^{-1}.$$

Let L be the lag operator, then note that:

$$(I - GL^{-1})(AF+B)(I-FL) = AL^{-1} + B + CL.$$

Thus, by the model's real determinacy, all of G's eigenvalues must also be inside

[39] The lack of terms in $\mathbb{E}_t\pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

the unit circle. Hence, since G and F all have all their eigenvalues in the unit circle, $(I - GL^{-1})$ and $(I - FL)$ are both invertible.

In terms of the lag operator, the equations determining z_t and $\tilde{z}_t$ are:

$$\begin{aligned}\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t &= -d\pi_t - \sum_{n=1}^{N} f_n \varepsilon_{n,t} \\ &= -d\tilde{\pi}_t - \sum_{n=1}^{N} f_n \varepsilon_{n,t} \\ &= \mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)\tilde{z}_t,\end{aligned}$$

as $\pi_t = \tilde{\pi}_t$ for all t. Consequently:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)(z_t - \tilde{z}_t) = 0.$$

Therefore, by the invertibility of $(I - GL^{-1})$, $(AF + B)$ and $(I - FL)$, $z_t = \tilde{z}_t$ for all t, as required. (Expectations drop out as the right-hand side is deterministic.)

The only slight difficulty with setting π_t^* as a function of structural shocks is that the central bank may struggle to observe these shocks. The central bank can certainly observe linear combinations of structural shocks, via estimating a VAR with sufficiently many lags. For variables that are plausibly contemporaneously exogenous, such as commodity prices for a small(ish) economy, this is already sufficient to recover the corresponding structural shock. To infer other shocks, the central bank needs to know more about the structure of the economy. However, we do not need to assume any more than is standard in rational expectations models. Forming rational expectations requires you to know the structure of the economy; if you know this structure, then you know the mapping from the reduced form shocks estimated by a VAR to the model's structural shocks.[40] Additionally, it is common to assume that the central bank responds to an output gap constructed by comparing outcomes to an economy without price rigidity. This already requires the central bank to know the values of all parameters and structural shocks.

K.7 Partially smoothed real rate rules

Suppose that the central bank sets interest rates according to the partially

[40] This mapping may not be unique valued if there are more shocks than observables. However, since we expect a relatively small number of shocks to explain the bulk of business cycle variance, this is unlikely to be problematic in practice.

smoothed real rate rule:

$$i_t - r_t = \varrho_i(i_{t-1} - r_{t-1}) + \mathbb{E}_t\pi^*_{t+1} - \varrho_i\mathbb{E}_{t-1}\pi^*_t + (1-\varrho_i)\phi(\pi_t - \pi^*_t),$$

where $\phi > 1$, $\varrho_i < 1$ and where π^*_t is the inflation target. Then, from the standard Fisher equation (without a wedge):

$$\mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1}) = \varrho_i\mathbb{E}_{t-1}(\pi_t - \pi^*_t) + (1-\varrho_i)\phi(\pi_t - \pi^*_t).$$

Now let $\hat{\pi}_t := \pi_t - \pi^*_t$ and $e_t := \mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1})$. Then we have the system:

$$e_t = \mathbb{E}_t\hat{\pi}_{t+1},$$

$$e_t = \varrho_i e_{t-1} + (1-\varrho_i)\phi\hat{\pi}_t.$$

Equivalently:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\mathbb{E}_t\begin{bmatrix} \hat{\pi}_{t+1} \\ e_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (1-\varrho_i)\phi & \varrho_i \end{bmatrix}\begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix},$$

so, from pre-multiplying by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

$$\mathbb{E}_t\begin{bmatrix} \hat{\pi}_{t+1} \\ e_t \end{bmatrix} = \begin{bmatrix} (1-\varrho_i)\phi & \varrho_i \\ (1-\varrho_i)\phi & \varrho_i \end{bmatrix}\begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix}.$$

Now:

$$\begin{bmatrix} (1-\varrho_i)\phi & \varrho_i \\ (1-\varrho_i)\phi & \varrho_i \end{bmatrix} = \begin{bmatrix} -\dfrac{\varrho_i}{(1-\varrho_i)\phi} & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \varrho_i + (1-\varrho_i)\phi \end{bmatrix}\begin{bmatrix} -\dfrac{\varrho_i}{(1-\varrho_i)\phi} & 1 \\ 1 & 1 \end{bmatrix}^{-1}.$$

Thus, if we define:

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} := \begin{bmatrix} -\dfrac{\varrho_i}{(1-\varrho_i)\phi} & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix} = \frac{(1-\varrho_i)\phi}{\varrho_i + (1-\varrho_i)\phi}\begin{bmatrix} -1 & 1 \\ 1 & \dfrac{\varrho_i}{(1-\varrho_i)\phi} \end{bmatrix}\begin{bmatrix} \hat{\pi}_t \\ e_{t-1} \end{bmatrix},$$

then:

$$\mathbb{E}_t\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \varrho_i + (1-\varrho_i)\phi \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

Now, since $\phi > 1$ and $\varrho_i < 1$, $\varrho_i + (1-\varrho_i)\phi = \phi - \varrho_i(\phi - 1) > 1$. Thus, the unique non-explosive solution for v_t is $v_t = 0$. (Note that v_t must be stationary as $\hat{\pi}_t$ and e_{t-1} must be stationary.) Hence, by the definition of v_t, $\hat{\pi}_t = -\frac{\varrho_i}{(1-\varrho_i)\phi}e_{t-1}$. So as $e_t = \mathbb{E}_t\hat{\pi}_{t+1}$, $e_t = -\frac{\varrho_i}{(1-\varrho_i)\phi}e_t$, i.e., $[\varrho_i + (1-\varrho_i)\phi]e_t = 0$, so $e_t = 0$, and hence $\hat{\pi}_t = 0$.

Therefore, with $\phi > 1$, $\pi_t = \pi^*_t$ is the unique stationary solution.

Finally, note that the coefficient on $\pi_t - \pi^*_t$ in the original rule was $(1-\varrho_i)\phi$, so for any $\theta > 0$ if we set $\phi := \frac{\theta}{1-\varrho_i}$ then for ϱ_i sufficiently close to 1, $\phi > 1$ as required. Thus, for ϱ_i sufficiently close to 1 a coefficient of $\theta > 0$ on $\pi_t - \pi^*_t$ will do. This links the results of this appendix to those of the main text.

K.8 Sunspot solutions under real rate rules

We are interested in sunspot solutions to the equations:

$$\pi_t - \pi^* = \beta \mathbb{E}_t(\pi_{t+1} - \pi^*) + \kappa x_t, \qquad x_t = \delta \mathbb{E}_t x_{t+1} - \varsigma(r_t - n),$$
$$\max\{0, r_t + \pi^* + \phi(\pi_t - \pi^*)\} = i_t = r_t + \mathbb{E}_t \pi_{t+1},$$

with $\kappa\varsigma \neq 0$, $\phi > 1$ and $n + \pi^* > 0$ which take the following form. While at the ZLB, there is a constant probability of $q_{\mathrm{Z}} \in [0,1]$ of remaining there in the next period. With probability $1 - q_{\mathrm{Z}}$ though, the economy escapes the ZLB. While nominal interest rates are positive, there is a constant probability of $q_{\mathrm{P}} \in [0,1]$ of remaining there in the next period. With probability $1 - q_{\mathrm{P}}$ though, the economy goes to the ZLB. Note that when $q_{\mathrm{P}} = 1$, the non-ZLB state is absorbing, so this two-state sunspot solution nests the absorbing case discussed in the main text.

We write r_{Z}, π_{Z} and x_{Z} for the values of r_t, π_t and x_t while at the ZLB, and r_{P}, π_{P} and x_{P} for the values of these variables when nominal interest rates are positive at i_{P}. Then, from the Euler equation and Phillips curve:

$$\begin{aligned}
x_{\mathrm{Z}} &= -\frac{\varsigma[\delta(1-q_{\mathrm{Z}})(r_{\mathrm{P}}-n) + (1-\delta q_{\mathrm{P}})(r_{\mathrm{Z}}-n)]}{(1-\delta)[1-\delta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]}, \\
x_{\mathrm{P}} &= -\frac{\varsigma[\delta(1-q_{\mathrm{P}})(r_{\mathrm{Z}}-n) + (1-\delta q_{\mathrm{Z}})(r_{\mathrm{P}}-n)]}{(1-\delta)[1-\delta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]}, \\
\pi_{\mathrm{Z}} - \pi^* &= \frac{\kappa[\beta(1-q_{\mathrm{Z}})x_{\mathrm{P}} + (1-\beta q_{\mathrm{P}})x_{\mathrm{Z}}]}{(1-\beta)[1-\beta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]} \\
&= -\frac{\kappa\varsigma[a_{\mathrm{ZZ}}(r_{\mathrm{Z}}-n) + a_{\mathrm{ZP}}(r_{\mathrm{P}}-n)]}{(1-\beta)(1-\delta)[1-\beta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)][1-\delta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]}, \\
\pi_{\mathrm{P}} - \pi^* &= \frac{\kappa[\beta(1-q_{\mathrm{P}})x_{\mathrm{Z}} + (1-\beta q_{\mathrm{Z}})x_{\mathrm{P}}]}{(1-\beta)[1-\beta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]} \\
&= -\frac{\kappa\varsigma[a_{\mathrm{PP}}(r_{\mathrm{P}}-n) + a_{\mathrm{PZ}}(r_{\mathrm{Z}}-n)]}{(1-\beta)(1-\delta)[1-\beta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)][1-\delta(q_{\mathrm{Z}}+q_{\mathrm{P}}-1)]},
\end{aligned}$$

where:

$$\begin{aligned}
a_{\mathrm{ZZ}} &:= \beta\delta(1-q_{\mathrm{Z}})(1-q_{\mathrm{P}}) + (1-\beta q_{\mathrm{P}})(1-\delta q_{\mathrm{P}}), \\
a_{\mathrm{ZP}} &:= (1-q_{\mathrm{Z}})[\beta + \delta - \beta\delta(q_{\mathrm{Z}}+q_{\mathrm{P}})], \\
a_{\mathrm{PP}} &:= \beta\delta(1-q_{\mathrm{Z}})(1-q_{\mathrm{P}}) + (1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}), \\
a_{\mathrm{PZ}} &:= (1-q_{\mathrm{P}})[\beta + \delta - \beta\delta(q_{\mathrm{Z}}+q_{\mathrm{P}})],
\end{aligned}$$

and from the Fisher equation and monetary rule:

$$0 = r_{\mathrm{Z}} + q_{\mathrm{Z}}\pi_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\pi_{\mathrm{P}} = r_{\mathrm{Z}} + \pi^* + q_{\mathrm{Z}}(\pi_{\mathrm{Z}} - \pi^*) + (1-q_{\mathrm{Z}})(\pi_{\mathrm{P}} - \pi^*),$$
$$\phi(\pi_{\mathrm{P}} - \pi^*) = q_{\mathrm{P}}(\pi_{\mathrm{P}} - \pi^*) + (1-q_{\mathrm{P}})(\pi_{\mathrm{Z}} - \pi^*).$$

So, if we define $\psi := \frac{1-q_{\mathrm{P}}}{\phi - q_{\mathrm{P}}} \in [0,1)$ and $b := \frac{\kappa\varsigma}{a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}}}$, then:

$$\pi_{\mathrm{Z}} - \pi^* = -b(r_{\mathrm{Z}} - n), \qquad \pi_{\mathrm{P}} - \pi^* = \psi(\pi_{\mathrm{Z}} - \pi^*) = -\psi b(r_{\mathrm{Z}} - n),$$
$$r_{\mathrm{P}} - n = \frac{\psi a_{\mathrm{ZZ}} - a_{\mathrm{PZ}}}{a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}}}(r_{\mathrm{Z}} - n),$$
$$r_{\mathrm{Z}} + \pi^* = -[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi](\pi_{\mathrm{Z}} - \pi^*) = b[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi](r_{\mathrm{Z}} - n),$$

as:

$$a_{\mathrm{ZZ}}a_{\mathrm{PP}} - a_{\mathrm{ZP}}a_{\mathrm{PZ}} = (1-\beta)(1-\delta)[1-\beta(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)][1-\delta(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)].$$

Thus:

$$r_{\mathrm{Z}} - n = -\frac{n+\pi^*}{1-[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]b},$$
$$r_{\mathrm{P}} - n = -\frac{\psi a_{\mathrm{ZZ}} - a_{\mathrm{PZ}}}{a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}}}\frac{n+\pi^*}{1-[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]b},$$
$$\pi_{\mathrm{Z}} - \pi^* = \frac{b(n+\pi^*)}{1-[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]b}, \qquad \pi_{\mathrm{P}} - \pi^* = \frac{\psi b(n+\pi^*)}{1-[q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]b}.$$

Our solution is consistent with equilibrium if and only if the monetary rule implies zero nominal rates in the Z state and positive nominal rates in the P state, i.e., if and only if:

$$0 \geq r_{\mathrm{Z}} + \pi^* + \phi(\pi_{\mathrm{Z}} - \pi^*) = \kappa\varsigma\frac{\phi - [q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]}{a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}} - [q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]\kappa\varsigma}(n+\pi^*),$$

and:

$$\begin{aligned} 0 &\leq r_{\mathrm{P}} + \pi^* + \phi(\pi_{\mathrm{P}} - \pi^*) \\ &= \frac{\psi(\phi-1)\kappa\varsigma + (1-\psi)(a_{\mathrm{PP}} + a_{\mathrm{PZ}} - q_{\mathrm{Z}}\kappa\varsigma)}{a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}} - [q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]\kappa\varsigma}(n+\pi^*), \end{aligned}$$

as $a_{\mathrm{PP}} + a_{\mathrm{PZ}} = a_{\mathrm{ZZ}} + a_{\mathrm{ZP}}$. For simplicity, suppose $\kappa\varsigma > 0$. Then, since $n + \pi^* > 0$ and $\phi > 1$ but $q_{\mathrm{Z}}, \psi \in [0,1]$, the two conditions hold if and only if $c_1 := a_{\mathrm{PP}} - \psi a_{\mathrm{ZP}} - [q_{\mathrm{Z}} + (1-q_{\mathrm{Z}})\psi]\kappa\varsigma \leq 0$ and $c_2 := \psi(\phi-1)\kappa\varsigma + (1-\psi)(a_{\mathrm{PP}} + a_{\mathrm{PZ}} - q_{\mathrm{Z}}\kappa\varsigma) \leq 0$. Now, note that as $\phi \to \infty$:

$$\begin{aligned} c_1 &\to q_{\mathrm{Z}}[(1-\beta)(1-\delta) - \kappa\varsigma] + (1-q_{\mathrm{Z}})[1-\beta\delta(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)], \\ c_2 &\to (q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)[(1-\beta)(1-\delta) - \kappa\varsigma] \\ &\qquad + [1-(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)][1-\beta\delta(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)]. \end{aligned}$$

Thus, at least when $(1-\beta)(1-\delta) - \kappa\varsigma < 0$ and $\beta\delta \geq 0$ (so $1 - \beta\delta(q_{\mathrm{Z}} + q_{\mathrm{P}} - 1)$ is decreasing in q_{Z} and q_{P}), for sufficiently high ϕ, the first condition holds if and only if q_{Z} is sufficiently high, and the second condition holds as well if and only if q_{P} is also sufficiently high.

Cleaner results (without assumptions on $\kappa\varsigma$ or $\beta\delta$, and without taking the limit as $\phi \to \infty$) are available in the absorbing case with $q_{\mathrm{P}} = 1$, in which case we have:

$$a_{\mathrm{ZZ}} = (1-\beta)(1-\delta), \qquad a_{\mathrm{ZP}} = (1-q_{\mathrm{Z}})[\beta + \delta - \beta\delta(q_{\mathrm{Z}}+1)],$$

$$a_{\mathrm{PP}} = (1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}), \qquad a_{\mathrm{PZ}} = 0, \qquad \psi = 0, \qquad b = \frac{\kappa\varsigma}{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}})},$$

$$r_{\mathrm{Z}} - n = -\frac{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}})(n+\pi^*)}{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}) - q_{\mathrm{Z}}\kappa\varsigma}, \qquad r_{\mathrm{P}} - n = 0,$$

$$\pi_{\mathrm{Z}} - \pi^* = \frac{\kappa\varsigma(n+\pi^*)}{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}) - q_{\mathrm{Z}}\kappa\varsigma}, \qquad \pi_{\mathrm{P}} - \pi^* = 0,$$

so the conditions become:

$$0 \geq (\phi - q_{\mathrm{Z}})\frac{\kappa\varsigma}{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}) - q_{\mathrm{Z}}\kappa\varsigma}(n+\pi^*),$$

and $0 \leq n + \pi^*$. Given that $\phi > 1 > q_{\mathrm{Z}}$, $\kappa\varsigma \neq 0$ and $n + \pi^* > 0$ by assumption, these hold if and only if:

$$\frac{(1-\beta q_{\mathrm{Z}})(1-\delta q_{\mathrm{Z}}) - q_{\mathrm{Z}}\kappa\varsigma}{\kappa\varsigma} \leq 0.$$

If this equality holds strictly, then by continuity, a sunspot solution must also exist for all $q_{\mathrm{P}} \in (1-\epsilon, 1]$, for some $\epsilon > 0$.

K.9 Perfect foresight uniqueness with the modified inflation target

Uniqueness conditional on the modified target. We want to prove uniqueness of equilibrium under the modified inflation target real rate rule of equations (12) and (13) of Subsection 4.2 (introduced in period 1), without uncertainty, and assuming that π_t and $\breve{\pi}_t^*$ are bounded in t, and that the economy eventually escapes the ZLB for good. The latter assumption implies there must exist a smallest possible $s \geq 1$ such that for all $t \geq s$, the ZLB does not bind. We assume for a contradiction that $s > 1$, hence for all $t \geq s$, by the monetary rule and Fisher equation:[41]

$$r_t + \pi_{t+1} = i_t = r_t + \pi_t + (\breve{\pi}_{t+1}^* - \breve{\pi}_t^*) + \theta(\pi_t - \breve{\pi}_t^*),$$

meaning:

$$\pi_{t+1} - \breve{\pi}_{t+1}^* = (1+\theta)(\pi_t - \breve{\pi}_t^*),$$

so for $t \geq s$, $\pi_t - \breve{\pi}_t^* = (1+\theta)^{t-s}(\pi_s - \breve{\pi}_s^*)$. Since $(1+\theta)^{t-s} \to \infty$ as $t \to \infty$, this in turn implies that $\pi_s = \breve{\pi}_s^*$, by our boundedness assumptions. But as the economy is at the ZLB in period $s-1$, $0 = i_{s-1} = r_{s-1} + \pi_s$, so $-r_{s-1} = \pi_s = \breve{\pi}_s^* \geq$

[41] Note that we can drop expectations as there is no uncertainty. It is OK to replace the lagged terms $i_{t-1} - r_{t-1} - \mathbb{E}_{t-1}\breve{\pi}_t^*$ with $\pi_t - \breve{\pi}_t^*$ as $t \geq s > 1$ so $t - 1 \geq 1$.

$\epsilon - r_{s-1} > -r_{s-1}$, giving the required contradiction. Thus $s = 1$, meaning the economy never hits the ZLB. Combined with the results of Subsection 2.1, this establishes the uniqueness of the $\pi_t = \breve{\pi}_t^* - \theta^{-1}\mathbb{E}_{t-1}(\pi_t - \breve{\pi}_t^*)$ solution conditional on the path of $\breve{\pi}_t^*$.

Uniqueness of the modified target. The only remaining source of potential multiplicity is the bound in the definition of $\breve{\pi}_t^*$, which may mean there are multiple possible paths of $\breve{\pi}_t^*$. Even if we assume that π_t^* is exogenous, r_t is not, so if r_t is sufficiently responsive to $\mathbb{E}_t\pi_{t+1}$, in theory there could be one solution in which $\pi_{t+1} = \breve{\pi}_{t+1}^* = \pi_{t+1}^* > -r_t + \epsilon$ and one solution in which $\pi_{t+1} = \breve{\pi}_{t+1}^* = -r_t + \epsilon > \pi_{t+1}^*$. However, this does not occur for standard models.

We illustrate this in the model given by equations (10) and (11), from Subsection 4.1. We assume that all exogenous processes (including π_t^*) are constant at their steady-state level, and that all variables are at steady-state in period 0, since neither assumption has any impact on uniqueness, by the results of Holden (2023). (This also means that our results are robust to adding any shocks to the model.) We also impose that the ZLB never binds, since we have already established this under our retained assumptions. Given this, we replace the notation $\breve{\pi}_{t+1}^*$ with $\breve{\pi}_{t+1|t}^*$, since $\breve{\pi}_{t+1}^*$ is known in period t given that π_t^* is now constant. Likewise, we replace π_{t+1} with $\pi_{t+1|t}$, as $\pi_{t+1} = \breve{\pi}_{t+1}^* = \breve{\pi}_{t+1|t}^*$, known at t. This gives the following equations for $t \geq 1$:

$$\beta(\pi_{t+1|t} - \pi^*) + \kappa x_t = \begin{cases} 0, & \text{if } t = 1 \\ \pi_{t|t-1} - \pi^*, & \text{if } t > 1 \end{cases}$$

$$i_t = \begin{cases} r_t + \breve{\pi}_{t+1|t}^*, & \text{if } t = 1 \\ r_t + \breve{\pi}_{t+1|t}^* + (1+\theta)\left(\pi_{t|t-1} - \breve{\pi}_{t|t-1}^*\right), & \text{if } t > 1' \end{cases}$$

$$x_t = \delta x_{t+1} - \varsigma(r_t - n), \qquad i_t = r_t + \pi_{t+1|t}, \qquad \breve{\pi}_{t+1|t}^* = \max\{\pi^*, \epsilon - r_t\},$$

where we assume $\kappa\varsigma \neq 0$, $\theta > 0$ and $n + \pi^* > \epsilon > 0$. The latter assumption ensures that $\breve{\pi}_{t|t-1}^* = \pi^*$ in steady state.

We are interested in the constraint in the definition of $\breve{\pi}_{t+1|t}^*$, which we note can be rewritten as the pair of equations:

$$z_t = \breve{\pi}_{t+1|t}^* + r_t - \epsilon, \qquad z_t = \max\{0, \pi^* + r_t - \epsilon\},$$

where z_t is an auxiliary variable. The results of Holden (2023) imply that in order to prove uniqueness under perfect foresight (conditional on z_t eventually converging to its positive steady state value), we should first replace the second equation for z_t just given with $z_t = \pi^* + r_t - \epsilon + y_t$, where y_t is an exogenous forcing process. For convenience, we define $y_0 := 0$. This implies that for $t \geq 1$:

$$\pi_{t+1|t} = \breve{\pi}^*_{t+1|t} = \pi^* + y_t,$$
$$x_t = \frac{1}{\kappa}(y_{t-1} - \beta y_t), \qquad r_t = n + \frac{1}{\kappa\varsigma}[-y_{t-1} + (\beta+\delta)y_t - \beta\delta y_{t+1}],$$
$$z_t = n + \pi^* - \epsilon + y_t + \frac{1}{\kappa\varsigma}[-y_{t-1} + (\beta+\delta)y_t - \beta\delta y_{t+1}],$$

from, respectively, the monetary rule and Fisher equation, the equations for z_t, the Phillips curve, the Euler equation, and the first equation for z_t.

Holden (2023) shows that uniqueness is determined by the determinants of the principal sub-matrices of the "M" matrix for the model, which, here, contains the partial derivatives of z_t (t in rows) with respect to y_s (s in columns). We take M to have infinitely many rows and columns in the following. By our solution for z_t, M is tridiagonal with $-\frac{1}{\kappa\varsigma}, 1 + \frac{\beta+\delta}{\kappa\varsigma}, -\frac{\beta\delta}{\kappa\varsigma}$ on the left, main and right diagonals respectively. We assume for now that $\frac{\beta+\delta}{\kappa\varsigma} \geq 0$.

Now consider a finite size principal sub-matrix of M. Since M is tridiagonal and Toeplitz, this sub-matrix must be block diagonal, where each block on the diagonal is either diagonal (with $1 + \frac{\beta+\delta}{\kappa\varsigma}$ on the diagonal), or tridiagonal (with $-\frac{1}{\kappa\varsigma}, 1 + \frac{\beta+\delta}{\kappa\varsigma}, -\frac{\beta\delta}{\kappa\varsigma}$ on the left, main and right diagonals respectively). Recall that the determinant of a block diagonal matrix is the product of the determinants of the blocks on the diagonal. Thus, the sub-matrix will have determinant greater or equal to one if each of the sub-matrix's blocks has determinant greater or equal to one. Since $\frac{\beta+\delta}{\kappa\varsigma} \geq 0$, a diagonal block of size $S \times S$ has determinant of $\left(1 + \frac{\beta+\delta}{\kappa\varsigma}\right)^S \geq 1$. Thus, we just need to check the determinants of the tridiagonal blocks.

Let:

$$d := \left(1 + \frac{\beta+\delta}{\kappa\varsigma}\right)^2 - 4\frac{\beta\delta}{(\kappa\varsigma)^2} = 1 + 2\frac{\beta+\delta}{\kappa\varsigma} + \frac{(\beta-\delta)^2}{(\kappa\varsigma)^2} \geq 1,$$

as we are assuming that $\frac{\beta+\delta}{\kappa\varsigma} \geq 0$. Then, by standard results on determinants of tridiagonal matrices,[42] the determinant of any $S \times S$ tridiagonal block is given by:

[42] This may be proven by using Laplace expansion twice to derive a recurrence for the determinant.

$$\frac{1}{2^{S+1}\sqrt{d}}\left[\left(1+\frac{\beta+\delta}{\kappa\varsigma}+\sqrt{d}\right)^{S+1}-\left(1+\frac{\beta+\delta}{\kappa\varsigma}-\sqrt{d}\right)^{S+1}\right]$$
$$=\frac{1}{2^S}\sum_{k=0}^{S+1}\binom{S+1}{k}\left(1+\frac{\beta+\delta}{\kappa\varsigma}\right)^k\sqrt{d}^{S-k}\frac{\left(1-(-1)^{S+1-k}\right)}{2}$$
$$\geq\frac{1}{2^S}\sum_{k=0}^{S+1}\binom{S+1}{k}\frac{\left(1-(-1)^{S+1-k}\right)}{2}=1.$$

Hence, the sub-matrix has determinant greater or equal to one. Thus, all principal minors of M are greater or equal to one, meaning that the M matrix is a "P-matrix" (Holden 2023), and moreover that no sufficiently small changes to the model could change this result.[43] (Being a P-matrix only requires positive principal minors, not ones greater or equal to one.) Thus, with π_t^* exogenous, the solution is robustly unique conditional on the terminal conditions (bounded inflation, eventual escapes from both bounds). For uniqueness without this additional robustness property, it is clearly sufficient that $d > 0$ and $1 + \frac{\beta+\delta}{\kappa\varsigma} > 0$, for example it is enough that $\frac{\beta+\delta}{\kappa\varsigma} > -\frac{1}{2}$.

K.10 Approximate uniqueness with endogenous wedges and multi-period bonds

In the set-up of Section 5, suppose we assume that Δ_t is stationary, and that there exists some $\overline{\mu}_0, \overline{\mu}_1, \overline{\mu}_2, \overline{\gamma}_0, \overline{\gamma}_1, \overline{\gamma}_2 \geq 0$ such that for any stationary solution for e_t, $|\mathbb{E}\Delta_t| \leq \overline{\mu}_0 + \overline{\mu}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \overline{\mu}_2 \operatorname{Var}(\pi_t - \pi_t^*)$ and $\operatorname{Var}\Delta_t \leq \overline{\gamma}_0 + \overline{\gamma}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \overline{\gamma}_2 \operatorname{Var}(\pi_t - \pi_t^*)$, for all $t \in \mathbb{Z}$ and $j, k \in \mathbb{N}$. This assumption is very mild, as discussed in Subsection 3.3 (and the form here is even milder, since it applies to Δ_t, not $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$). Now note that:

$$|\mathbb{E}(\pi_t - \pi_t^*)| = \frac{1}{\theta}|\mathbb{E}(e_t - e_{t-1} + \Delta_t)| \leq \frac{1}{\theta}[2|\mathbb{E}e_t| + |\mathbb{E}\Delta_t|],$$

(by the triangle inequality and stationarity) and:

[43] This robustness holds for any fixed size M matrix. I.e., fix $T > 0$ (potentially extremely large) and suppose the bound ceases to apply more than T periods in the future. Then following a sufficiently small change to the model, there will be a unique solution that satisfies the bound for T periods, but which may violate it after T periods.

$$\begin{aligned}\operatorname{Var}(\pi_t - \pi_t^*) &= \frac{1}{\theta^2}\operatorname{Var}(e_t - e_{t-1} + \Delta_t) \\ &\le \frac{1}{\theta^2}\left[4\operatorname{Var} e_t + 4\sqrt{(\operatorname{Var} e_t)(\operatorname{Var}\Delta_t)} + \operatorname{Var}\Delta_t\right] \\ &\le \frac{1}{\theta^2}[8\operatorname{Var} e_t + 2\operatorname{Var}\Delta_t],\end{aligned}$$

(by Cauchy-Schwarz, stationarity and the fact that for all $z \ge 0$, $4\sqrt{(\operatorname{Var} e_t)z} \le 4\operatorname{Var} e_t + z$). Thus, if θ is large enough to ensure $\theta > \overline{\mu}_1$, $\theta^2 > 2\overline{\gamma}_2$ and $(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) > 2\overline{\mu}_2\overline{\gamma}_1$, then:

$$|\mathbb{E}\Delta_t| \le \frac{(\theta^2 - 2\overline{\gamma}_2)(\theta^2\overline{\mu}_0 + 2\theta\overline{\mu}_1|\mathbb{E}e_t| + 8\overline{\mu}_2\operatorname{Var} e_t) + 2\overline{\mu}_2(\theta^2\overline{\gamma}_0 + 2\theta\overline{\gamma}_1|\mathbb{E}e_t| + 8\overline{\gamma}_2\operatorname{Var} e_t)}{\theta[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]},$$

and:

$$\operatorname{Var}\Delta_t \le \frac{\overline{\gamma}_1(\theta^2\overline{\mu}_0 + 2\theta\overline{\mu}_1|\mathbb{E}e_t| + 8\overline{\mu}_2\operatorname{Var} e_t) + (\theta - \overline{\mu}_1)(\theta^2\overline{\gamma}_0 + 2\theta\overline{\gamma}_1|\mathbb{E}e_t| + 8\overline{\gamma}_2\operatorname{Var} e_t)}{(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1}.$$

Therefore, as $e_t = \mathbb{E}_t \sum_{j=1}^{\infty}(1+\theta T)^{-\left\lceil \frac{j}{T-L+S}\right\rceil}\Delta_{t+j}$:

$$\begin{aligned}&|\mathbb{E}e_t| \le \sum_{j=1}^{\infty}(1+\theta T)^{-\left\lceil \frac{j}{T-L+S}\right\rceil}|\mathbb{E}\Delta_t| \\ &= (T-L+S)\sum_{j=1}^{\infty}(1+\theta T)^{-j}|\mathbb{E}\Delta_t| = \frac{T-L+S}{\theta T}|\mathbb{E}\Delta_t| \le \frac{|\mathbb{E}\Delta_t|}{\theta} \\ &\le \frac{(\theta^2 - 2\overline{\gamma}_2)(\theta^2\overline{\mu}_0 + 2\theta\overline{\mu}_1|\mathbb{E}e_t| + 8\overline{\mu}_2\operatorname{Var} e_t) + 2\overline{\mu}_2(\theta^2\overline{\gamma}_0 + 2\theta\overline{\gamma}_1|\mathbb{E}e_t| + 8\overline{\gamma}_2\operatorname{Var} e_t)}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]},\end{aligned}$$

(using the triangle inequality and stationarity), and:

$$\begin{aligned}&\operatorname{Var} e_t = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(1+\theta T)^{-\left\lceil \frac{j}{T-L+S}\right\rceil}(1+\theta T)^{-\left\lceil \frac{k}{T-L+S}\right\rceil}\operatorname{Cov}(\mathbb{E}_t\Delta_{t+j}, \mathbb{E}_t\Delta_{t+k}) \\ &\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(1+\theta T)^{-\left\lceil \frac{j}{T-L+S}\right\rceil}(1+\theta T)^{-\left\lceil \frac{k}{T-L+S}\right\rceil}\operatorname{Var}\Delta_t \\ &= \left[(T-L+S)\sum_{j=1}^{\infty}(1+\theta T)^{-j}\right]^2\operatorname{Var}\Delta_t = \left(\frac{T-L+S}{\theta T}\right)^2\operatorname{Var}\Delta_t \le \frac{\operatorname{Var}\Delta_t}{\theta^2} \\ &\le \frac{\overline{\gamma}_1(\theta^2\overline{\mu}_0 + 2\theta\overline{\mu}_1|\mathbb{E}e_t| + 8\overline{\mu}_2\operatorname{Var} e_t) + (\theta - \overline{\mu}_1)(\theta^2\overline{\gamma}_0 + 2\theta\overline{\gamma}_1|\mathbb{E}e_t| + 8\overline{\gamma}_2\operatorname{Var} e_t)}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]}\end{aligned}$$

(by the inequality for covariances of conditional expectations derived in Subsection 3.3). Now, define:

$$\mu_0 := \frac{\theta^2(\theta^2 - 2\overline{\gamma}_2)\overline{\mu}_0 + 2\theta^2\overline{\mu}_2\overline{\gamma}_0}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta}\right) \text{ as } \theta \to \infty,$$

$$
\begin{aligned}
\mu_1 &:= \frac{2\theta(\theta^2 - 2\overline{\gamma}_2)\overline{\mu}_1 + 4\theta\overline{\mu}_2\overline{\gamma}_1}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta^2}\right) \text{ as } \theta \to \infty,\\
\mu_2 &:= \frac{8(\theta^2 - 2\overline{\gamma}_2)\overline{\mu}_2 + 16\overline{\mu}_2\overline{\gamma}_2}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta^3}\right) \text{ as } \theta \to \infty,\\
\gamma_0 &:= \frac{\theta^2\overline{\gamma}_1\overline{\mu}_0 + \theta^2(\theta - \overline{\mu}_1)\overline{\gamma}_0}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta^2}\right) \text{ as } \theta \to \infty,\\
\gamma_1 &:= \frac{2\theta\overline{\gamma}_1\overline{\mu}_1 + 2\theta(\theta - \overline{\mu}_1)\overline{\gamma}_1}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta^3}\right) \text{ as } \theta \to \infty,\\
\gamma_2 &:= \frac{8\overline{\gamma}_1\overline{\mu}_2 + 8(\theta - \overline{\mu}_1)\overline{\gamma}_2}{\theta^2[(\theta - \overline{\mu}_1)(\theta^2 - 2\overline{\gamma}_2) - 2\overline{\mu}_2\overline{\gamma}_1]} = \mathrm{O}\left(\frac{1}{\theta^4}\right) \text{ as } \theta \to \infty,
\end{aligned}
$$

then we can rewrite the previous inequalities as:

$$
\begin{aligned}
|\mathbb{E}\Delta_t| &\leq \theta\mu_0 + \theta\mu_1|\mathbb{E}e_t| + \theta\mu_2 \operatorname{Var} e_t,\\
\operatorname{Var}\Delta_t &\leq \theta^2\gamma_0 + \theta^2\gamma_1|\mathbb{E}e_t| + \theta^2\gamma_2 \operatorname{Var} e_t,\\
|\mathbb{E}e_t| &\leq \mu_0 + \mu_1|\mathbb{E}e_t| + \mu_2 \operatorname{Var} e_t,\\
\operatorname{Var} e_t &\leq \gamma_0 + \gamma_1|\mathbb{E}e_t| + \gamma_2 \operatorname{Var} e_t,
\end{aligned}
$$

Now, suppose that θ is large enough that additionally $1 > \mu_1$, $1 > \gamma_1$ and $(1 - \mu_1)(1 - \gamma_2) > \mu_2\gamma_1$ (note that these inequalities always hold for sufficiently large θ, by the previously derived big-O asymptotics), then:

$$
\begin{aligned}
|\mathbb{E}e_t| &\leq \frac{(1 - \gamma_2)\mu_0 + \mu_2\gamma_0}{(1 - \mu_1)(1 - \gamma_2) - \mu_2\gamma_1} = \mathrm{O}\left(\frac{1}{\theta}\right) \text{ as } \theta \to \infty,\\
\operatorname{Var} e_t &\leq \frac{(1 - \mu_1)\gamma_0 + \gamma_1\mu_0}{(1 - \mu_1)(1 - \gamma_2) - \mu_2\gamma_1} = \mathrm{O}\left(\frac{1}{\theta^2}\right) \text{ as } \theta \to \infty,
\end{aligned}
$$

by the previously derived big-O asymptotics. Hence, as $\theta \to \infty$, $\mathbb{E}e_t \to 0$ and $\operatorname{Var} e_t \to 0$, as required.

Finally, note that by the bounds on $|\mathbb{E}(\pi_t - \pi_t^*)|$ and $\operatorname{Var}(\pi_t - \pi_t^*)$ derived above, we have that:

$$
\begin{aligned}
|\mathbb{E}(\pi_t - \pi_t^*)| &\leq \frac{1}{\theta}[2|\mathbb{E}e_t| + |\mathbb{E}\Delta_t|] \leq \frac{2}{\theta}|\mathbb{E}e_t| + \mu_0 + \mu_1|\mathbb{E}e_t| + \mu_2 \operatorname{Var} e_t\\
&= \mathrm{O}\left(\frac{1}{\theta}\right) \text{ as } \theta \to \infty,
\end{aligned}
$$

and:

$$
\begin{aligned}
\operatorname{Var}(\pi_t - \pi_t^*) &\leq \frac{1}{\theta^2}[8 \operatorname{Var} e_t + 2 \operatorname{Var}\Delta_t]\\
&\leq 8\frac{1}{\theta^2}\operatorname{Var} e_t + 2\gamma_0 + 2\gamma_1|\mathbb{E}e_t| + 2\gamma_2 \operatorname{Var} e_t = \mathrm{O}\left(\frac{1}{\theta^2}\right) \text{ as } \theta \to \infty,
\end{aligned}
$$

so, as required, $\mathbb{E}(\pi_t - \pi_t^*) \to 0$ and $\operatorname{Var}(\pi_t - \pi_t^*) \to 0$ as $\theta \to \infty$.

K.11 Convergence under least squares learning

We have that a_t and b_t are updated according to the recursion:

$$\begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} a_{t-1} \\ b_{t-1} \end{bmatrix} + \frac{1}{t+w}\frac{1}{v}\begin{bmatrix} v \\ \zeta_t \end{bmatrix}(\pi_t - a_{t-1} - b_{t-1}\zeta_t),$$

where:

$$\pi_t = \frac{1}{\phi - m_t}[(1-m_t)a_{t-1} + (\rho - m_t)b_{t-1}\zeta_t - \zeta_t],$$

and:

$$m_t = \frac{1}{t+w}\left(1 + \rho\frac{\zeta_t^2}{v}\right).$$

Note:

$$\pi_t = \frac{a_{t-1} + (\rho b_{t-1} - 1)\zeta_t}{\phi} - \frac{m_t}{\phi - m_t}\frac{(\phi-1)a_{t-1} + (\phi-\rho)b_{t-1}\zeta_t + \zeta_t}{\phi}.$$

Now define:

$$\mathcal{H}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta\right) = -\frac{1}{\phi v}[(\phi-1)a + (\phi-\rho)b\zeta + \zeta]\begin{bmatrix} v \\ \zeta \end{bmatrix},$$

$$\mathcal{R}_t\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta\right) = -\frac{1 + \rho\frac{\zeta^2}{v}}{\phi - \frac{1}{t+w}\left(1 + \rho\frac{\zeta^2}{v}\right)}\frac{1}{\phi v}[(\phi-1)a + (\phi-\rho)b\zeta + \zeta]\begin{bmatrix} v \\ \zeta \end{bmatrix}.$$

We now verify each of the parts of assumption D.1 of Subsection 6.7 of Evans & Honkapohja (2001). We assume that $\phi > 1$.

Part (i):

$$\begin{aligned}
&\mathcal{H}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta_1\right) - \mathcal{H}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta_2\right) \\
&\qquad = -\frac{1}{\phi v}\left[(\phi-1)a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + [(\phi-\rho)b + 1]\begin{bmatrix} v \\ \zeta_1 + \zeta_2 \end{bmatrix}\right](\zeta_1 - \zeta_2).
\end{aligned}$$

So:

$$\begin{aligned}
&\left\|\mathcal{H}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta_1\right) - \mathcal{H}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \zeta_2\right)\right\|_2 \\
&\qquad \le \frac{1}{\phi v}[(\phi-1)|a| + [(\phi-\rho)|b| + 1](v + |\zeta_1| + |\zeta_2|)]|\zeta_1 - \zeta_2| \\
&\qquad \le \frac{\max\{1,v\}}{\phi v}[(\phi-1)|a| + (\phi-\rho)|b| + 1]|\zeta_1 - \zeta_2|(1 + |\zeta_1| + |\zeta_2|) \\
&\qquad \le \sqrt{2}\frac{(\phi-\rho)\max\{1,v\}}{\phi v}\left[1 + \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2\right]|\zeta_1 - \zeta_2|(1 + |\zeta_1| + |\zeta_2|).
\end{aligned}$$

Part (ii):

$$\mathcal{H}\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, 0\right) - \mathcal{H}\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, 0\right) = -\frac{\phi - 1}{\phi v}(a_1 - a_2)\begin{bmatrix} v \\ 0 \end{bmatrix}.$$

So:

$$\left\|\mathcal{H}\left(\begin{bmatrix}a_1\\b_1\end{bmatrix},0\right)-\mathcal{H}\left(\begin{bmatrix}a_2\\b_2\end{bmatrix},0\right)\right\|_2=\frac{\phi-1}{\phi}|a_1-a_2|\le\frac{\phi-1}{\phi}\left\|\begin{bmatrix}a_1\\b_1\end{bmatrix}-\begin{bmatrix}a_2\\b_2\end{bmatrix}\right\|_2.$$

Part (iii):

$$\begin{aligned}&\frac{\partial}{\partial\zeta}\mathcal{H}\left(\begin{bmatrix}a_1\\b_1\end{bmatrix},\zeta\right)-\frac{\partial}{\partial\zeta}\mathcal{H}\left(\begin{bmatrix}a_2\\b_2\end{bmatrix},\zeta\right)\\&\qquad=-\frac{1}{\phi v}\left[(\phi-1)(a_1-a_2)\begin{bmatrix}0\\1\end{bmatrix}+(\phi-\rho)(b_1-b_2)\begin{bmatrix}v\\2\zeta\end{bmatrix}\right].\end{aligned}$$

So:

$$\begin{aligned}&\left\|\frac{\partial}{\partial\zeta}\mathcal{H}\left(\begin{bmatrix}a_1\\b_1\end{bmatrix},\zeta\right)-\frac{\partial}{\partial\zeta}\mathcal{H}\left(\begin{bmatrix}a_2\\b_2\end{bmatrix},\zeta\right)\right\|_2\\&\qquad\le\frac{1}{\phi v}\left[(\phi-1)|a_1-a_2|+(\phi-\rho)|b_1-b_2|(v+2|\zeta|)\right]\\&\qquad\le\frac{(\phi-\rho)\max\{2,v\}}{\phi v}\left[|a_1-a_2|+|b_1-b_2|\right](1+|\zeta|)\\&\qquad\le\sqrt{2}\frac{(\phi-\rho)\max\{2,v\}}{\phi v}\left\|\begin{bmatrix}a_1\\b_1\end{bmatrix}-\begin{bmatrix}a_2\\b_2\end{bmatrix}\right\|_2(1+|\zeta|).\end{aligned}$$

Part (iv):

Let Φ be the cumulative distribution function of the standard normal distribution. Then:

$$\Pr\left(\frac{1}{t+w}\left(1+|\rho|\frac{\zeta_t^2}{v}\right)>1\right)=2\Phi\left(-\sqrt{\frac{t+w-1}{|\rho|}}\right).$$

Thus:

$$\begin{aligned}\sum_{t=1}^{\infty}\Pr\left(\frac{1}{t+w}\left(1+|\rho|\frac{\zeta_t^2}{v}\right)>1\right)&=2\Phi\left(-\sqrt{\frac{w}{|\rho|}}\right)+\sum_{t=2}^{\infty}2\Phi\left(-\sqrt{\frac{t+w-1}{|\rho|}}\right)\\&\le2\Phi\left(-\sqrt{\frac{w}{|\rho|}}\right)+\int_1^\infty2\Phi\left(-\sqrt{\frac{t+w-1}{|\rho|}}\right)\mathrm{d}t\\&=\sqrt{\frac{2|\rho|w}{\bar{\pi}}}\exp\left(-\frac{w}{2|\rho|}\right)+2(1-w+|\rho|)\Phi\left(-\sqrt{\frac{w}{|\rho|}}\right)<\infty\end{aligned}$$

where $\bar{\pi}$ is the mathematical constant usually denoted by π. Hence, by the Borel-Cantelli lemma, with probability one $\frac{1}{t+w}\left(1+|\rho|\frac{\zeta_t^2}{v}\right)>1$ for only finitely many t.

When $\frac{1}{t+w}\left(1+|\rho|\frac{\zeta^2}{v}\right)\le1$, we have that:

$$\begin{aligned}\left\|\mathcal{R}_t\left(\begin{bmatrix}a\\b\end{bmatrix},\zeta\right)\right\|&\le\frac{1+|\rho|\frac{\zeta^2}{v}}{\phi-1}\frac{(\phi-1)|a|+(\phi-\rho)|b||\zeta|+|\zeta|}{\phi v}(v+|\zeta|)\\&\le\frac{\max\{1,v\}}{\phi v}\max\left\{1,\frac{|\rho|}{v}\right\}\frac{\phi-\rho}{\phi-1}(1+|a|+|b|)(1+|\zeta|)^2(1+|\zeta|^2)\\&\le4\sqrt{2}\frac{\max\{1,v\}}{\phi v}\max\left\{1,\frac{|\rho|}{v}\right\}\frac{\phi-\rho}{\phi-1}\left(1+\left\|\begin{bmatrix}a\\b\end{bmatrix}\right\|_2\right)(1+|\zeta|^4).\end{aligned}$$

If this held without assuming $\frac{1}{t+w}\left(1+|\rho|\frac{\zeta^2}{v}\right) \leq 1$, then this would satisfy Part (iv) of Assumption D.1 of Section 6.7 of Evans & Honkapohja (2001). However, from inspecting the proof of Theorem 6.10 of Evans & Honkapohja (2001), contained in the proof of Theorem 17 of Section 1.9 of Benveniste, Métivier & Priouret (1990), a weaker assumption is sufficient. In fact, it is enough that there exists $C_2 > 0$ and $q > 0$ such that for all $a, b \in \mathbb{R}$:[44]

$$\Pr\left(\exists T \text{ s.t. } \forall t \geq T, \left\|\mathcal{R}_t\left(\begin{bmatrix}a\\b\end{bmatrix}, \zeta_t\right)\right\| \leq C_2\left(1+\left\|\begin{bmatrix}a\\b\end{bmatrix}\right\|_2\right)(1+|\zeta_t|^q)\right) = 1.$$

This is satisfied, by our result that with probability one $\frac{1}{t+w}\left(1+|\rho|\frac{\zeta_t^2}{v}\right) > 1$ for only finitely many t.

This completes the verification of Assumption D.1 of Section 6.7 of Evans & Honkapohja (2001). Assumption D.2 trivially holds, as ζ_t is a stationary AR(1) process. Assumption A.1 also clearly holds.

Now define:

$$\begin{aligned} h\left(\begin{bmatrix}a\\b\end{bmatrix}\right) &= \lim_{t\to\infty} \mathbb{E}\mathcal{H}\left(\begin{bmatrix}a\\b\end{bmatrix}, \zeta_t\right) = -\frac{1}{\phi v}\mathbb{E}\begin{bmatrix}[(\phi-1)a+(\phi-\rho)b\zeta_t+\zeta_t]v\\(\phi-1)a\zeta_t+(\phi-\rho)b\zeta_t^2+\zeta_t^2\end{bmatrix} \\ &= -\frac{1}{\phi}\begin{bmatrix}(\phi-1)a\\1+(\phi-\rho)b\end{bmatrix}. \end{aligned}$$

Then, the ordinary differential equation (ODE):

$$\frac{\mathrm{d}\begin{bmatrix}a(\tau)\\b(\tau)\end{bmatrix}}{\mathrm{d}\tau} = h\left(\begin{bmatrix}a(\tau)\\b(\tau)\end{bmatrix}\right),$$

has the unique solution $a(\tau) = a(0)\exp\left(-\frac{\phi-1}{\phi}\tau\right)$, $b(\tau) = -\frac{1}{\phi-\rho} + \left(b_0 + \frac{1}{\phi-\rho}\right)\exp\left(-\frac{\phi-\rho}{\phi}\tau\right)$, which converges to the unique equilibrium point $a = 0$ and $b = -\frac{1}{\phi-\rho}$ as $\tau \to \infty$, since $\phi > 1 > \rho$. Now define $\mathcal{U}\colon \mathbb{R}^2 \to \mathbb{R}$ by:

$$\mathcal{U}\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = a^2 + \left(b + \frac{1}{\phi-\rho}\right)^2.$$

Clearly $\mathcal{U}$ is non-negative and twice continuously differentiable. We now verify $\mathcal{U}$ satisfies the other conditions of Theorem 6.10 of Evans & Honkapohja (2001).

Part (i):

$$\frac{\partial\mathcal{U}\left(\begin{bmatrix}a\\b\end{bmatrix}\right)}{\partial\begin{bmatrix}a\\b\end{bmatrix}} h\left(\begin{bmatrix}a(\tau)\\b(\tau)\end{bmatrix}\right) = -2\frac{\phi-1}{\phi}a^2 - 2\frac{\phi-\rho}{\phi}\left(b+\frac{1}{\phi-\rho}\right)^2 \leq 0,$$

[44] Note that the condition in equation (1.9.2) of Benveniste, Métivier & Priouret (1990) is only used in the proof of Lemma 18 of Section 1.9.

(using numerator layout notation for the derivative), with equality if and only if $a = 0$ and $b = -\frac{1}{\phi - \rho}$.

Part (ii):

$\mathcal{U}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if and only if $a = 0$ and $b = -\frac{1}{\phi - \rho}$.

Part (iii):

Suppose $\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2 \geq \frac{2}{\phi - \rho}$. Then $a^2 + b^2 \geq \frac{4}{(\phi - \rho)^2}$, so:

$$\begin{aligned}
\mathcal{U}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) - \frac{1}{4}\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2 &= \frac{3}{4}(a^2 + b^2) + \frac{2b}{\phi - \rho} + \frac{1}{(\phi - \rho)^2} \\
&\geq \frac{3}{4}(a^2 + b^2) - \frac{2\left(\frac{1}{\phi - \rho} + \frac{\phi - \rho}{4}(a^2 + b^2)\right)}{\phi - \rho} + \frac{1}{(\phi - \rho)^2} \\
&= \frac{1}{4}(a^2 + b^2) - \frac{1}{(\phi - \rho)^2} \geq 0.
\end{aligned}$$

This completes the verification of the conditions of Theorem 6.10 of Evans & Honkapohja (2001). Hence, with probability one, a_t converges to 0 and b_t converges to $-\frac{1}{\phi - \rho}$.

K.12 Optimal consumption with perpetuities and a permanent ZLB

For the sake of illustration, we adopt the simple parametric set-up used in Online Appendix G.1. It is clear our results are not specific to this set-up, however.

We suppose the representative household supplies one unit of labour, inelastically. Production of the final good is given by $y_t = l_t (= 1)$. In period t, the representative household maximises $\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \log c_{t+k}$, subject to the budget constraint:

$$P_t c_t + A_t + Q_t B_t + P_t \tau_t = P_t y_t + I_{t-1} A_{t-1} + B_{t-1}(1 + \omega Q_t),$$

where c_t is consumption, τ_t are real lump sum taxes, P_t is the price of the final good, A_t is the number of one period nominal bonds purchased by the household at t, which each return I_t in period $t + 1$, Q_t is the price of a long (geometric coupon) bond and B_t are the number of units of this long bond purchased by the household at t. One unit of the period t long bond bought at t returns \$1 at $t + 1$, along with $\omega \in (0,1]$ units of the period $t + 1$ bond.

The household first order conditions imply:

$$1 = \beta I_t \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}}, \qquad Q_t = \beta \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}} (1 + \omega Q_{t+1}).$$

The household transversality conditions are that:

$$\lim_{k\to\infty}\beta^k\mathbb{E}_t\frac{A_{t+k}}{P_{t+k}c_{t+k}}=0,\qquad \lim_{k\to\infty}\beta^k\mathbb{E}_t\frac{Q_{t+k}B_{t+k}}{P_{t+k}c_{t+k}}=0,$$

but we do not assume the second is necessary when $\omega=1$. (The necessity of the transversality constraint when $\omega<1$ follows from the following test given in Kamihigashi (2006), and formally proven in Kamihigashi (2003): "Shift the entire optimal path [for the state variable] downward by a small fixed proportion. Does it reduce the value of the objective function by only a finite amount? If so, the transversality condition is necessary.")

The government issues no one period bonds, so $A_t=0$. The government fixes the supply of long-bonds at $B_t=B_t^*:=B_{-1}\omega^{t+1}$. The central bank pegs nominal interest rates at the ZLB, meaning $I_t=1$.

The final goods market clears, so $y_t=c_t=1$. Thus, from the household budget constraint, we have the following government budget constraint:

$$Q_tB_t^*+P_t\tau_t=B_{t-1}^*(1+\omega Q_t).$$

We assume that the government adjusts taxes τ_t period by period to ensure this always holds (i.e., fiscal policy is passive and Ricardian). Thus, $P_t\tau_t=B_{-1}\omega^t$.

Let $\Pi_t:=\frac{P_t}{P_{t-1}}$, then from market clearing and the Euler equation for nominal bonds, $1=\beta\mathbb{E}_t\frac{1}{\Pi_{t+1}}$. So, from the Euler equation for the long bond:

$$Q_t=\frac{1}{1-\omega}+\lim_{k\to\infty}\omega^k\beta^k\mathbb{E}_t\left[\prod_{j=1}^{k}\frac{1}{\Pi_{t+j}}\right]Q_{t+k}\geq\frac{1}{1-\omega},$$

with equality when $\omega<1$ as the transversality constraint definitely holds in that case. But, when $\omega=1$, this says $Q_t\geq\infty$, so $Q_t=\infty$, hence $Q_t=\frac{1}{1-\omega}$ for all $\omega\in[0,1]$. Now let $b_t:=\frac{Q_tB_t}{P_t}$, then from the budget constraint:

$$P_tc_t+P_tb_t+P_t\tau_t=P_t+\frac{P_{t-1}b_{t-1}}{Q_{t-1}}(1+\omega Q_t)=P_t+P_{t-1}b_{t-1},$$

and thus:

$$c_t+b_t+\tau_t=1+\frac{b_{t-1}}{\Pi_t}.$$

It is instructive to re-solve the original household problem under this rewritten budget constraint. This must have the same solution as the original problem. In particular, consider the problem of maximising $\mathbb{E}_t\sum_{k=0}^{\infty}\beta^k\log c_{t+k}$, subject to:

$$c_t+b_t+\tau_t=1+\frac{b_{t-1}}{\Pi_t},$$

by choosing $c_t, c_{t+1}, \dots, b_t, b_{t+1}, \dots$. This is the "textbook" cake eating problem with exogenous income, $1 - \tau_t$, and gross interest rate $\frac{1}{\Pi_t}$. The Euler equation is $\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1} c_{t+1}}$, and the (always necessary) transversality constraint states that $\lim_{k\to\infty} \beta^k \mathbb{E}_t \frac{b_{t+k}}{c_{t+k}} = 0$.

Additionally, the government budget constraint can be rewritten as:

$$\tau_t = (1-\omega) b_{-1} \left[\prod_{s=0}^{t} \frac{1}{\Pi_s} \right] \omega^t.$$

We know that in equilibrium, market clearing implies $c_t = 1$, but for now, we will "forget" this fact, and merely suppose that $c_t = c$ for all t, for some $c > 0$. This satisfies the Euler equation as:

$$\frac{1}{c} = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1} c} = \frac{1}{c},$$

as $1 = \beta \mathbb{E}_t \frac{1}{\Pi_{t+1}}$. Then transversality simplifies to $\lim_{k\to\infty} \beta^k \mathbb{E}_t b_{t+k} = 0$, and the budget constraint gives:

$$\begin{aligned} b_t &= \sum_{k=0}^{t} \left[\prod_{j=0}^{k-1} \frac{1}{\Pi_{t-j}} \right] (1 - c_{t-k} - \tau_{t-k}) + \left[\prod_{j=0}^{t} \frac{1}{\Pi_{t-j}} \right] b_{-1} \\ &= (1-c) \sum_{k=0}^{t} \prod_{s=t-k+1}^{t} \frac{1}{\Pi_s} + \omega^{t+1} b_{-1} \prod_{s=0}^{t} \frac{1}{\Pi_s}, \end{aligned}$$

by the simplified government budget constraint previously derived. Hence, since $1 = \mathbb{E}_t \beta \frac{1}{\Pi_{t+1}}$:

$$\beta^t \mathbb{E}_0 b_t = (1-c) \frac{1-\beta^{t+1}}{1-\beta} + \omega^{t+1} b_{-1} \frac{1}{\Pi_0},$$

so, by the period 0 transversality constraint:

$$0 = \lim_{t\to\infty} \beta^t \mathbb{E}_0 b_t = \frac{1-c}{1-\beta} + b_{-1} \frac{1}{\Pi_0} \lim_{t\to\infty} \omega^{t+1}.$$

If $\omega \in (0,1)$, then this implies that $c = 1$ as expected. However, if $\omega = 1$, then:

$$c = 1 + (1-\beta) \frac{b_{-1}}{\Pi_0}.$$

Thus, if Π_0 is finite, then $c > 1$, violating the market clearing condition. The only way to restore market clearing is if Π_0 is infinite. This is intuitive, as when $\omega = 1$, households have infinite nominal wealth, which cannot fail to push up prices.

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